Faculté de physique et ingénierie
Université de Strasbourg

University of Strasbourg

Tutorial I

J. Polonyi, M. Dufour , S. Whitlock

Transcribed by
Pierre Guichard

M1-S1 2022

Reminder

EXERCISE I : Hermitian operators

1. A and B are two Hermitian operators. Under which condition is the operator $C = AB$ also Hermitian ?

$$A = A^\dagger \qquad\qquad B = B^\dagger$$

We want to have

$$C = C^\dagger$$

Meaning that

$$C^\dagger = (AB)^\dagger = B^\dagger A^\dagger = BA$$

Hence,

$$AB = BA$$

Meaning that the commutator $[A, B] = 0$.

2. Show that the eigenvalues of a Hermitian operator A are real and that the corresponding eigenvectors are orthogonal.

$|\psi\rangle$ is an eigenvector of A with eigenvalue λ.

$$A|\psi\rangle = \lambda\psi \qquad \langle\psi| A^\dagger = \lambda^* \langle\psi| \qquad \langle\psi| A = \lambda^* \langle\psi|$$

Hence,

$$\langle\psi| A |\psi\rangle = \lambda \langle\psi|\psi\rangle \qquad \langle\psi| A |\psi\rangle = \lambda^* \langle\psi|\psi\rangle$$

Meaning that

$$\lambda = \lambda^*$$

which is true for $\lambda \in \mathbb{R}$.

$$A|\psi_1\rangle = \lambda_1 |\psi_1\rangle \qquad A|\psi_2\rangle = \lambda_2 |\psi_2\rangle$$

$$\langle\psi_2| A |\psi_1\rangle = \lambda_1 \langle\psi_2|\psi_1\rangle \qquad \langle\psi_1| A |\psi_2\rangle = \lambda_2 \langle\psi_1|\psi_2\rangle$$

$$\langle\psi_2| A |\psi_1\rangle^* = \langle\psi_1| A^\dagger |\psi_2\rangle = \langle\psi_1| A |\psi_2\rangle$$

Hence we can write,

$$\langle\psi_2| A |\psi_1\rangle - \langle\psi_1| A |\psi_2\rangle = (\lambda_1 - \lambda_2) \langle\psi_1|\psi_2\rangle = 0$$

We assumed $\lambda_1 \neq \lambda_2$, implying that $\langle\psi_1|\psi_2\rangle = 0$.

3. Show that if two observables A and B commute and if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigenvectors of A with different eigenvalues, then $\langle\psi_1| B |\psi_2\rangle = 0$. This implies if two observables A and B commute, we can build an orthonormal basis from the space of the states consisting of common eigenvectors to A and B.

$$\langle\psi_1| [A, B] |\psi_2\rangle = \langle\psi_1| AB - BA |\psi_2\rangle = 0$$

$$\lambda_1 \langle\psi_1| B |\psi_2\rangle - \lambda_2 \langle\psi_1| B |\psi_2\rangle = 0$$

Meaning that

$$\langle\psi_1| B |\psi_2\rangle (\lambda_1 - \lambda_2) = 0$$

We assumed $\lambda_1 \neq \lambda_2$, implying that $\langle\psi_1| B |\psi_2\rangle = 0$.
Assume $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenvectors of A and form a basis of state space

$$\lambda_1 \neq \lambda_2 \text{ since } \langle\psi_1| B |\psi_2\rangle = 0$$

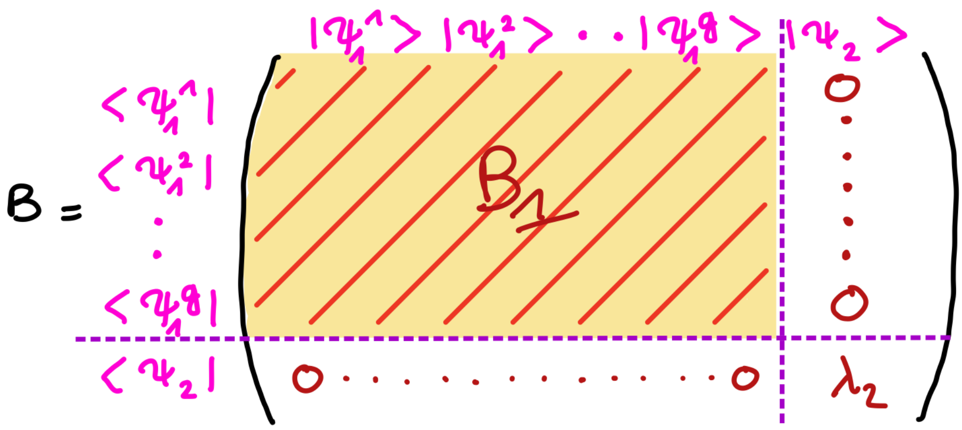
B is diagonal in this basis

$|\psi_1\rangle$, $|\psi_2\rangle$ are also eigenvectors of B

In case of degenerate eigenvalues, by definition we can write, (assuming λ_1 is degenerate),

$$A \left|\psi_1^j\right\rangle = \lambda_1 \left|\psi_1^j\right\rangle$$

where $j \in \mathbb{N}$ is the degeneracy.



B_1 is the restriction of B to the subspace $\mathcal{E}_1$. This matrix is a block-diagonal matrix. One can always diagonalize B_1 and find a new basis $\{\left|\psi_1^j\right\rangle\}$ in such a way that

$$\left\langle\psi_1^i\right| B \left|\psi_1^j\right\rangle \propto \delta_{ij}$$

4. A_1, A_2 and H are three Hermitian operators. We assume that $[A_1, H] = 0$ and $[A_2, H] = 0$. Explain why if $[A_1, A_2] \neq 0$, then H has at least one degenerate eigenvalue

   Proof by contradiction : we assume that all eigenvalues of H are non-degenerate and $[A_1, A_2] \neq 0$, since $[A_1, H] = 0$ and $[A_2, H] = 0$, then basically all eigenvectors of H are also eigenvectors of A_1 and all eigenvectors of H are also eigenvectors of A_2, meaning that all vectors of A_1 are vectors of A_2, hence $[A_1, A_2] = 0$, there is a contradiction, then there is at least one degenerate eigenvalue.

EXERCISE II : Expectation values

Let A be an observable. We denote $|\psi(t)\rangle$ the wavefunction of the system at time t. The expectation value of the observable A is then defined as $\langle A(t)\rangle = \langle\psi(t)| A |\psi(t)\rangle$.

1. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle = \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{i\hbar}\langle [A, H]\rangle$$

where H is the Hamiltonian of the system.

Let A be an observable represented by the autoadjoint operator $\hat{A}$. We define its mean value by

$$\langle \hat{A}\rangle = \langle\psi(t)| \hat{A} |\psi(t)\rangle$$

We derivate its equality by the time :

$$\frac{\mathrm{d}\langle \hat{A}\rangle}{\mathrm{d}t} = \frac{\partial \langle\psi(t)|}{\partial t}|\hat{A}|\psi(t)\rangle + \langle\psi(t)| \frac{\partial \hat{A}}{\partial t}|\psi(t)\rangle + \langle\psi(t)| \hat{A}|\frac{\partial |\psi(t)\rangle}{\partial t}$$

We then use Schrödinger's equation and its conjugate

$$+i\hbar\frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = \hat{H}\,|\psi(t)\rangle \qquad \text{and} \qquad -i\hbar\frac{\mathrm{d}\,\langle\psi(t)|}{\mathrm{d}t} = \langle\psi(t)|\,\hat{H}$$

And we replace in the previous equation, we get

$$\frac{\mathrm{d}\langle \hat{A}\rangle}{\mathrm{d}t} = \langle\frac{\partial \hat{A}}{\partial t}\rangle + \frac{1}{i\hbar}\langle\psi(t)|\,\hat{A}\hat{H}\,|\psi(t)\rangle - \frac{1}{i\hbar}\langle\psi(t)|\,\hat{H}\hat{A}\,|\psi(t)\rangle$$

With $[\hat{A}, \hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A}$, we finally get

$$\frac{\mathrm{d}\langle \hat{A}\rangle}{\mathrm{d}t} = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{1}{i\hbar}\left\langle [\hat{A}, \hat{H}]\right\rangle$$

2. We want to apply the previous result to the observables x and p for a spinless particle described by the Hamiltonian $H = p^2/2m+V(x)$ where $V(x)$ is a scalar potential. Establish the equations of motion for the expectation values $\langle x(t)\rangle$ and $\langle p(t)\rangle$ and compare the obtained result to Hamilton's equations from classical mechanics.

- $\langle \hat{x}(t)\rangle$:

$$\begin{aligned}
[\hat{x}, \hat{H}] &= \hat{x}\frac{\hat{p}^2}{2m} + \hat{x}\hat{V}(\hat{x}) - \frac{\hat{p}^2}{2m}\hat{x} - \hat{V}(\hat{x})\hat{x} \\
&= \hat{x}\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\hat{x} \\
&= \left[\hat{x}, \frac{\hat{p}^2}{2m}\right]
\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle &= \langle\frac{\partial\hat{x}}{\partial t}\rangle + \frac{1}{i\hbar}\langle[\hat{x},\hat{H}]\rangle \\ &= \frac{1}{i\hbar}\langle[\hat{x},\frac{\hat{p}^2}{2m}]\rangle \\ &= \frac{1}{i\hbar 2m}\langle\hat{p}[\hat{x},\hat{p}] + [\hat{x},\hat{p}]\hat{p}\rangle\end{aligned}$$

We do remind that $[\hat{x},\hat{p}] = i\hbar$:

$$\begin{aligned}\frac{1}{i\hbar 2m}\langle\hat{p}[\hat{x},\hat{p}] + [\hat{x},\hat{p}]\hat{p}\rangle &= \frac{1}{i\hbar 2m}\langle 2i\hbar\hat{p}\rangle \\ &= \frac{1}{m}\langle\hat{p}\rangle\end{aligned}$$

- $\langle\hat{p}(t)\rangle$:

$$\begin{aligned}[\hat{p},\hat{H}] &= \hat{p}\frac{\hat{p}^2}{2m} + \hat{p}\hat{U}(\hat{x}) - \frac{\hat{p}^2}{2m}\hat{p} - \hat{U}(\hat{x})\hat{p} \\ &= \hat{p}\hat{U}(\hat{x}) - \hat{U}(\hat{x})\hat{p} \\ &= \left[\hat{p},\hat{U}(\hat{x})\right]\end{aligned}$$

We know that $\hat{p} = -i\hbar\nabla$

$$\left[\hat{p},\hat{U}(\hat{x})\right] = -i\hbar\nabla\hat{U}(\hat{x}) + i\hbar\hat{U}(\hat{x})\nabla$$

Where,

$$\begin{aligned}\left\langle\left[\hat{p},\hat{U}(\hat{x})\right]\right\rangle &= -i\hbar\left\langle\psi\middle|\nabla(\hat{U}\psi)\right\rangle + i\hbar\left\langle\psi\middle|\hat{U}(\nabla\psi)\right\rangle \\ &= -i\hbar\left\langle\psi\middle|\hat{U}(\nabla\psi)\right\rangle - i\hbar\left\langle\psi\middle|(\nabla\hat{U})\psi\right\rangle + i\hbar\left\langle\psi\middle|\hat{U}(\nabla\psi)\right\rangle \\ &= -i\hbar\left\langle\psi\middle|(\nabla\hat{U})\psi\right\rangle \\ &= i\hbar\langle F\rangle\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle &= \frac{1}{i\hbar}\langle -i\hbar\nabla\hat{U}(\hat{x})\rangle \\ &= \langle F\rangle\end{aligned}$$

Here we recognize Hamilton's canonical equations applied to mean quantities. It suffices to differentiate the first with respect to time to find Newton's second law.
In most practical cases, one has

$$\frac{\partial}{\partial x}V(\langle x\rangle) = \left\langle\frac{\partial}{\partial x}V(x)\right\rangle$$

in harmonic oscillator, or at the macroscopic limit where the characteristic distance (De Broglie wavelength) $\ll$ the typical distance over which $V(x)$ varies.
Those findings are really important on the correspondance principle

EXERCISE III : Uncertainty relation

Consider two observables A and B.

1. By writing the commutator of A and B in the form $[A, B] = iC$, show that C is Hermitian.

$$A = A^\dagger \qquad\qquad B = B^\dagger$$

$$C = -i[A, B] \quad C^\dagger = i[A, B]^\dagger = i(AB - BA)^\dagger = i(B^\dagger A^\dagger - A^\dagger B^\dagger) = i[B, A] = -i[A, B] = C$$

Meaning that C is hermitian

2. Prove the Heisenberg uncertainty relation :

$$(\Delta A)^2 (\Delta B)^2 \geq -\frac{1}{4}\langle [A, B]\rangle^2$$

where the variance $(\Delta X)^2 = \langle (X - \langle X\rangle)^2\rangle = \langle \delta X^2\rangle$, for $X = A$ or B.
Hint : consider the vector $|\varphi\rangle = (\delta A + i\lambda\delta B)|\psi\rangle$ and compute the dot product $\langle\varphi|\varphi\rangle$, where λ is any real constant ($|\lambda|^2 \geq 0$).

$\delta A = \delta A^\dagger$ and $\delta B = \delta B^\dagger$.

$$\langle\varphi|\varphi\rangle \geq 0 \Longleftrightarrow \langle\psi|(\delta A^\dagger - i\lambda\delta B^\dagger)(\delta A + i\lambda\delta B)|\psi\rangle \geq 0$$
$$\langle\psi|\delta A^2|\psi\rangle + i\lambda\langle\psi|[\delta A, \delta B]|\psi\rangle + \lambda^2\langle\psi|\delta^2 B|\psi\rangle \geq 0$$
$$\langle\psi|\delta A^2|\psi\rangle - \lambda\langle\psi|C|\psi\rangle + \lambda^2\langle\psi|\delta B^2|/psi\rangle \geq 0$$

How do we find the lambda that minimize it.
It is a polynomial in λ, so we take the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\langle\delta A^2\rangle - \lambda\langle C\rangle + \lambda^2\langle\delta B^2\rangle\right) = 0$$

We find,

$$\lambda = \frac{\langle C\rangle}{2\langle\delta B^2\rangle}$$

Thus we get

$$\langle\delta A^2\rangle - \frac{\langle C\rangle^2}{2\langle\delta B^2\rangle^2} + \frac{\langle C\rangle^2}{4\langle\delta B^2\rangle} \geq 0$$

Meaning,

$$\langle\delta A^2\rangle\langle\delta B^2\rangle \geq \frac{\langle C\rangle^2}{4}$$

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}\left[-i\langle [A, B]\rangle\right]^2$$

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}\left[\langle [A, B]\rangle\right]^2$$

This is the *uncertainty principle.*

EXERCISE IV : Baker-Campbell-Hausdorff formula

The Baker-Campbell-Hausdorff (BCH) formula is an expansion for the product of exponential functions that depend on non-commuting operators A and B :

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots} \tag{1}$$

It is especially useful in studying the time evolution of quantum systems and in quantum optics. In this exercise we prove a special case of the BCH formula :

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{2}$$

valid when $[A,[A,B]] = [B,[A,B]] = 0$, which implies all terms in Eq. 1 from the third order upwards are zero.

1. Show that $[B, A^n] = nA^{n-1}[B, A]$, e.g., using mathematical induction.
   Hint : You might find the following identity useful : $[B, A^k] = [B, A]A^{k-1} + A[B, A^{k-1}]$

   $$[B, A^n] = nA^{n-1}[B, A]$$

   rank $n = 1$ is trivially true, suppose that property holds at rank n,

   $$\begin{aligned} [B, A^{n+1}] &= (n+1)A^n[B, A] \\ &= [B, A]A^n + A[B, A^n] \\ &= [B, A]A^n + A\{nA^{n-1}[B, A]\} \\ &= [B, A]A^n + nA^n[B, A] \\ &= A^n[B, A] + nA^n[B, A] \\ &= (n+1)A^n[B, A] \end{aligned}$$

   hence it is true at rank $n + 1$, so it is true for rank n.

2. Show that $[B, e^{-Ax}] = -xe^{-Ax}[B, A]$, where x is a constant.

   $$e^{-Ax} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} A^n x^n$$

$$
\begin{aligned}
[B, e^{-Ax}] &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!}[B, A^n]x^n \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} n A^{n-1}[B, A]x^n \\
&= \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} n A^{n-1}[B, A]x^n \\
&= -x \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(n-1)!} A^{n-1}x^{n-1}[B, A] \\
&= -x \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} A^m x^m [B, A] \\
&= -xe^{-Ax}[B, A]
\end{aligned}
$$

3. We suppose $f(x) = e^{Ax}e^{Bx}$. Evaluate $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ as a function of A, B, $[A, B]$ and $f(x)$.

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= Ae^{Ax}e^{Bx} + e^{Ax}Be^{Bx} \\
&= (A + e^{Ax}B^{-Ax})e^{Ax}e^{Bx} \\
&= (A + e^{Ax}B^{-Ax})ef(x) \\
&= A + e^{Ax}(e^{-Ax}B - xe^{-Ax}[B, A])f(x) \\
&= (A + B + x[A, B])f(x)
\end{aligned}
$$

4. By integrating the obtained differential equation, establish the relationship given by equation 2.

$$
\frac{\mathrm{d}f(x)}{f(x)}(A + B + x[A, B])\mathrm{d}x
$$

$$
\ln(f(x)) = Ax + Bx + \frac{x^2}{2}[A, B] + \text{constant}
$$

$$
f(x) = f(0)e^{xA + xB + \frac{x^2}{2}[A,B]}
$$

we take $x = 1$ and we get the BCH formula.

EXERCISE V : Temporal evolution of a two-level system

Consider the Hamiltonian H of a two-level system represented in matrix form :

$$H = \hbar \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

in the basis $\left\{ |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

The eigenvalues of H are given by $E_\pm = \pm\hbar\sqrt{a^2+b^2}$ and the corresponding eigenvectors are

$$|\chi_+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle \qquad \text{and} \qquad |\chi_-\rangle = \cos\frac{\theta}{2}|\downarrow\rangle - \sin\frac{\theta}{2}|\uparrow\rangle\, s$$

with $\cos\theta = a/\sqrt{a^2+b^2}$, $\sin\theta = b/\sqrt{a^2+b^2}$, $\tan\theta = b/a$.

1. The state vector $|\varphi(t)\rangle$ can be decomposed in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ in the form

$$|\varphi(t)\rangle = c_\uparrow(t)|\uparrow\rangle + c_\downarrow(t)|\downarrow\rangle$$

Write the system of coupled differential equations for the coefficients $c_\uparrow(t)$ and $c_\downarrow(t)$.

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\varphi(t)\rangle = H|\varphi(t)\rangle$$

we take $\hbar = 1$,

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_\uparrow(t) \\ c_\downarrow(t) \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} c_\uparrow(t) \\ c_\downarrow(t) \end{pmatrix}$$

$$\begin{cases} i\dfrac{\mathrm{d}}{\mathrm{d}t}c_\uparrow(t) = ac_\uparrow(t) + bc_\downarrow(t) \\ i\dfrac{\mathrm{d}}{\mathrm{d}t}c_\downarrow(t) = bc_\uparrow(t) - ac_\downarrow(t) \end{cases} \qquad (\star)$$

2. Show that $|\varphi(t)\rangle = \lambda e^{-i\Omega t/2}|\chi_+\rangle + \mu e^{i\Omega t/2}|\chi_-\rangle$ is a general solution to these equations, where λ and μ are time independent constants with $|\lambda^2+\mu^2| = 1$ and $\hbar\Omega = 2\hbar\sqrt{a^2+b^2}$ is the energy difference $E_+ - E_-$.

$$\begin{aligned} |\varphi(t)\rangle &= \lambda e^{-i\Omega t/2}|\chi_+\rangle + \mu e^{i\Omega t/2}|\chi_-\rangle \\ &= \lambda e^{-i\Omega t/2}\left(\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle\right) + \mu e^{i\Omega t/2}\left(-\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle\right) \end{aligned}$$

we can identify $c_\uparrow(t)$ and $c_\downarrow(t)$,

$$\begin{cases} c_\uparrow(t) = \lambda e^{-i\Omega t/2}\cos\left(\frac{\theta}{2}\right) - \mu e^{i\Omega t/2}\sin\left(\frac{\theta}{2}\right) \\ c_\downarrow(t) = \lambda e^{-i\Omega t/2}\sin\left(\frac{\theta}{2}\right) + \mu e^{i\Omega t/2}\cos\left(\frac{\theta}{2}\right) \end{cases}$$

$$\begin{aligned} i\frac{\partial}{\partial t}c_\uparrow(t) &= i\lambda\left(-\frac{i\Omega}{2}\right)e^{-i\Omega t/2}\cos\left(\frac{\theta}{2}\right) - i\mu\left(\frac{i\Omega}{2}\right)e^{i\Omega t/2}\sin\left(\frac{\theta}{2}\right) \\ &= \frac{\lambda\Omega}{2}\cos\left(\frac{\theta}{2}\right)e^{-i\Omega t/2} + \frac{\mu\Omega}{2}\sin\left(\frac{\theta}{2}\right)e^{i\Omega t/2} \end{aligned}$$

We use ($\star$) to identify

$$i\frac{\partial}{\partial t}c_\uparrow(t) = \lambda\left[a\cos\left(\frac{\theta}{2}\right) + b\sin\left(\frac{\theta}{2}\right)\right]e^{-i\Omega t/2} + \mu\left[b\cos\left(\frac{\theta}{2}\right) - a\sin\left(\frac{\theta}{2}\right)\right]e^{i\Omega t/2}$$

3. Suppose $c_\uparrow(t=0) = 0$. Deduce λ and μ (within any possible phase factors) and derive an expression for the probability to find the system at time t in the state $|\uparrow\rangle$.

$$\begin{cases} c_\uparrow(0) = \lambda\cos\left(\frac{\theta}{2}\right) - \mu\sin\left(\frac{\theta}{2}\right) = 0 \\ |\lambda|^2 + |\mu|^2 = 1 \end{cases}$$

So a possible solution is

$$\lambda = \sin\left(\frac{\theta}{2}\right) \qquad\qquad \mu = \cos\left(\frac{\theta}{2}\right)$$

So we get,

$$\begin{aligned} c_\uparrow(t) &= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i\Omega t/2} - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{i\Omega t/2} \\ &= 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(\frac{e^{-i\Omega t/2} - e^{+i\Omega t/2}}{2i}\right) \\ &= -2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\Omega t}{2}\right) \\ &= -i\sin(\theta)\sin\left(\frac{\Omega t}{2}\right) \end{aligned}$$

And for $c_\downarrow(t)$,

$$c_\downarrow(t) = \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\Omega t/2} + \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{i\Omega t/2}$$

So, we get

$$P_\uparrow(t) = |\langle\uparrow|\varphi(t)\rangle|^2 = |c_\uparrow(t)|^2 = \sin^2(\theta)\sin^2\left(\frac{\Omega t}{2}\right) = \frac{b^2}{a^2+b^2}\sin^2\left(\frac{\Omega t}{2}\right)$$

4. What is the corresponding expression if $c_\uparrow(t=0)=1$?

$$\begin{cases} c_\uparrow(t=0) = \lambda\cos\left(\dfrac{\theta}{2}\right) - \mu\sin\left(\dfrac{\theta}{2}\right) = 1 \\ |\lambda|^2 + |\mu|^2 = 1 \end{cases}$$

A possible solution is

$$\lambda = \cos\left(\frac{\theta}{2}\right) \qquad\qquad \mu = -\sin\left(\frac{\theta}{2}\right)$$

$$\begin{aligned} c_\uparrow(t) &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i\Omega t/2} + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{+i\Omega t/2} \\ &= \cos^2\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta t}{2}\right) - i\sin\left(\frac{\Omega t}{2}\right)\right] + \sin^2\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\Omega t}{2}\right) + i\sin\left(\frac{\Omega t}{2}\right)\right] \\ &= \cos\left(\frac{\Omega t}{2}\right) - i\left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]\sin\left(\frac{\Omega t}{2}\right) \\ &= \cos\left(\frac{\Omega t}{2}\right) - i\cos(\theta)\sin\left(\frac{\Omega t}{2}\right) \end{aligned}$$

We can calculate,

$$\begin{aligned} P_\uparrow(t) = |c_\uparrow(t)|^2 &= \cos^2\left(\frac{\Omega t}{2}\right) + \cos^2(\theta)\sin^2\left(\frac{\Omega t}{2}\right) \\ &= 1 - \sin^2\left(\frac{\Omega t}{2}\right) + \cos^2(\theta)\sin^2\left(\frac{\Omega t}{2}\right) \\ &= \sin^2\left(\frac{\Omega t}{2}\right)\left(1-\cos^2(\theta)\right) + 1 \\ &= 1 - \sin^2(\theta)\sin^2\left(\frac{\Omega t}{2}\right) \end{aligned}$$

Since

$$P_\uparrow(t) + P_\downarrow(t) = 1$$

We can get the expression,

$$P_\downarrow(t) = \sin^2(\theta)\sin^2\left(\frac{\Omega t}{2}\right)$$