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# Tutorial IV

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# Reminder

# EXERCISE 0 : Reminder, spin operators, matrix form

Write the state $|\uparrow\rangle$ in the $|\leftarrow\rangle$, $|\rightarrow\rangle$ basis, where $S_x |\rightarrow\rangle = +\frac{\hbar}{2} |\rightarrow\rangle$ and $S_x |\leftarrow\rangle = -\frac{\hbar}{2} |\leftarrow\rangle$.

For a spin $S = 1/2$,

$$(S_x) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (S_y) = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (S_z) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From the past sheet of exercises,

$$b'_n = \sum_k \langle \phi'_n | \phi_k \rangle \, b_k$$

$$\phi'_n = \{|\rightarrow\rangle, |\leftarrow\rangle\} \qquad \phi_k = \{|\uparrow\rangle, |\downarrow\rangle\}$$

# EXERCISE I : Coupling of two spin-$1/2$ particles

We consider a system of two independent spin-1/2 particles. In this exercise, only the spin degree of freedom is taken into account. We denote $|m_1, m_2\rangle$ as the tensor product of the states $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$, which are eigenvectors associated with the complete set of commuting observables $\{s_1^2, s_{1z}\}$ and $\{s_2^2, s_{2z}\}$. We denote the total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$.

1. Write down the matrix representation of the operators $S_z$ and $S^2$ in the basis of $\{|m_1, m_2\rangle\}$. *Hint : it might be helpful to expand $S^2$ in terms of the raising and lowering operators for each spin $s_{1\pm}$,$s_{2\pm}$*

$$(\hat{S}_z) = \begin{pmatrix} \langle\uparrow\uparrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\uparrow\uparrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\uparrow\uparrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\uparrow\uparrow|\hat{S}_z|\downarrow\downarrow\rangle \\ \langle\uparrow\downarrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\uparrow\downarrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\uparrow\downarrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\uparrow\downarrow|\hat{S}_z|\downarrow\downarrow\rangle \\ \langle\downarrow\uparrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\downarrow\uparrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\downarrow\uparrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\downarrow\uparrow|\hat{S}_z|\downarrow\downarrow\rangle \\ \langle\downarrow\downarrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\downarrow\downarrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\downarrow\downarrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\downarrow\downarrow|\hat{S}_z|\downarrow\downarrow\rangle \end{pmatrix}$$

Because of the orthogonality, $(\hat{S}_z)$ is diagonal,

$$\begin{cases} \hat{S}_z|\uparrow\uparrow\rangle = (\hat{s}_{1z} + \hat{s}_{2z})|\uparrow\uparrow\rangle = \hbar\left(\frac{1}{2} + \frac{1}{2}\right)|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle \\ \hat{S}_z|\uparrow\downarrow\rangle = (\hat{s}_{1z} + \hat{s}_{2z})|\uparrow\downarrow\rangle = \hbar\left(\frac{1}{2} - \frac{1}{2}\right)|\uparrow\downarrow\rangle = 0|\uparrow\downarrow\rangle \\ \hat{S}_z|\downarrow\uparrow\rangle = (\hat{s}_{1z} + \hat{s}_{2z})|\downarrow\uparrow\rangle = \hbar\left(-\frac{1}{2} + \frac{1}{2}\right)|\downarrow\uparrow\rangle = 0|\downarrow\uparrow\rangle \\ \hat{S}_z|\downarrow\downarrow\rangle = (\hat{s}_{1z} + \hat{s}_{2z})|\downarrow\downarrow\rangle = \hbar\left(-\frac{1}{2} - \frac{1}{2}\right)|\downarrow\downarrow\rangle = -\hbar|\downarrow\downarrow\rangle \end{cases}$$

Meaning,

$$(\hat{S}_z) = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We can express $(\hat{S}^2)$,

$$(\hat{S}^2) = (\hat{s}_1 + \hat{s}_2)^2 = \hat{s}_1^2 + \hat{s}_2^2 + 2\vec{s}_1 \cdot \vec{s}_2$$

$$\begin{cases} \hat{s}_1^2|m_1m_2\rangle = \hbar^2 s_1(s_1+1)|m_1m_2\rangle = \frac{3\hbar^2}{4}|m_1m_2\rangle \\ \hat{s}_2^2|m_1m_2\rangle = \hbar^2 s_2(s_2+1)|m_1m_2\rangle = \frac{3\hbar^2}{4}|m_1m_2\rangle \end{cases}$$

Meaning,

$$(\hat{s}_1^2 + \hat{s}_2^2) = \frac{3\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And now, we want to express $2\vec{s}_1 \cdot \vec{s}_2$ with the ladders operators,

$$\begin{cases} \hat{s}_{1x} = \frac{1}{2}(\hat{s}_{1+} + \hat{s}_{1-}) \\ \hat{s}_{1y} = \frac{1}{2}(\hat{s}_{1+} - \hat{s}_{1-}) \end{cases}$$

$$2\vec{s}_1 \cdot \vec{s}_2 = \underbrace{(s_{1+}s_{2-} + s_{1-}s_{2+})}_{\text{flip-flop term}} + 2s_{1z}s_{2z}$$

$$\begin{cases} (\hat{s}_{1+}\hat{s}_{2-} + \hat{s}_{1-}\hat{s}_{2+})\left|\uparrow\uparrow\right\rangle = 0 \\ (\hat{s}_{1+}\hat{s}_{2-} + \hat{s}_{1-}\hat{s}_{2+})\left|\uparrow\downarrow\right\rangle = \hbar^2\left|\downarrow\uparrow\right\rangle \\ (\hat{s}_{1+}\hat{s}_{2-} + \hat{s}_{1-}\hat{s}_{2+})\left|\downarrow\uparrow\right\rangle = \hbar^2\left|\uparrow\downarrow\right\rangle \\ (\hat{s}_{1+}\hat{s}_{2-} + \hat{s}_{1-}\hat{s}_{2+})\left|\downarrow\downarrow\right\rangle = 0 \end{cases}$$

Meaning,

$$\text{flip-flop} = \hbar^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2\hat{s}_{1z}\hat{s}_{2z} = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Meaning,

$$(\hat{S}^2) = \frac{3\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \hbar^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

2. What are the eigenvalues and eigenvectors of $S_z$ ?

$$(\hat{S}_z) = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$(\hat{S}_z)$ is already diagonal in this basis, meaning

$$\lambda = \{+1, 0, 0, -1\}$$

Of eignstates,

$$\begin{cases} |\uparrow\uparrow\rangle \longrightarrow \lambda = +1 \\ |\downarrow\downarrow\rangle \longrightarrow \lambda = -1 \\ |\uparrow\downarrow\rangle \, , \; |\downarrow\uparrow\rangle \longrightarrow \lambda = 0 \end{cases}$$

3. What are the eigenvalues and eigenvectors of $S^2$ ?

$$(\hat{S}^2) = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\det(\hat{S}^2 - \lambda \hat{1}) = 0$$

$$\det \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2((1-\lambda)^2 - 1) = 0$$

$$\lambda = \{2, 2, 2, 0\}$$

Eigenvalue equation,

$$(A - \lambda \hat{1})\psi = 0$$

For $\lambda = 2$,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

Meaning,

$$\begin{bmatrix} \left\{ a = d = 0, b = c = \frac{1}{\sqrt{2}} \right\} \longrightarrow \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ \{a = 1, b = c = d = 0\} \longrightarrow |\uparrow\uparrow\rangle \\ \{a = 0, b = c = 0, d = 1\} \longrightarrow |\downarrow\downarrow\rangle \end{bmatrix} \text{ these are triplet states}$$

For $\lambda = 0$,

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

Meaning,

$$\left\{ a = 0, b = \frac{1}{\sqrt{2}}, c = -\frac{1}{\sqrt{2}}, d = 0 \right\} \longrightarrow \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \text{ this is a singlet state (only one of them)}$$

4. Find a set of common eigenstates to both $S^2$ and $S_z$ expressed using the basis of $\{|m_1, m_2\rangle\}$.

$$\left.\begin{matrix} |\uparrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{matrix}\right\} |S, M\rangle \left\{ \begin{matrix} |11\rangle \\ |1-1\rangle \\ |10\rangle \\ |00\rangle \end{matrix}\right.$$

# EXERCISE II : Coupling of two angular momenta $l = 1$

We consider a system of two particles of orbital angular momenta $l_1 = l_2 = 1$. We denote $|L, M\rangle$ the eigenstates associated with the total angular momentum of the system.

1. What are the possible values of $L$ ? For a given value of $L$, what are the possible values of $M$ ? Count the states $|L, M\rangle$ and compare with the size of the state spaces for each particle.

$$|l_1 - l_2| \leq L \leq |l_1 + l_2| \qquad\qquad -L \leq M \leq +L$$

Meaning there is three possible values for $L$, $L \in \{0, 1, 2\}$.
There is $2L + 1$ states for $M$. $L = 0$, $M \in \{0\}$. $L = 1$, $M \in \{0, \pm 1\}$. $L = 2$, $M \in \{0, \pm 1, \pm 2\}$.
Meaning, there is a total of 9 states.
For a single particle, $|l, m\rangle$,

$$|1, 0\rangle \qquad\qquad |1, -1\rangle \qquad\qquad |1, +1\rangle$$

Meaning we have, the exponential scaling of the Hilbert Space, $(2l + 1)^N$.

2. Give the expression for the state $|L = 2, M = 2\rangle$ a deduce from it expressions for the other angular momentum states $|L, M\rangle$ using the basis of states $|l_1 = 1, l_2 = 1, m_1, m_2\rangle$.

$|L = 2, M = 2\rangle$ in the basis $|l_1 = 1, m_1, l_2 = 1, m_2\rangle$

$$L_- |L, M\rangle = \hbar\sqrt{L(L+1) - M(M-1)}\,|L, M-1\rangle$$

Find a representation in the uncoupled basis of $|L = 2, M = 1\rangle$,

$$L_- |L = 2, M = 1\rangle = 2\hbar\,|2, 1\rangle \iff (l_{1-} \oplus l_{2-})\,|1, 1, 1, 1\rangle = \sqrt{2}\hbar(|1, 0, 1, 1\rangle + |1, 1, 1, 0\rangle)$$

So we get,

$$|2, 1\rangle = \frac{1}{\sqrt{2}}(|1, 0, 1, 1\rangle + |1, 1, 1, 0\rangle)$$

$$|2, 0\rangle = \frac{1}{\sqrt{6}}(|1, -1, 1, 1\rangle + 2\,|1, 0, 1, 0\rangle + |1, 1, 1, -1\rangle)$$

# EXERCISE III : Properties of Clebsch-Gordan coefficients

1. Using $|J, M\rangle = \sum_{m_1,m_2} \langle j_1, m_1, j_2, m_2|J, M\rangle\, |j_1, m_1, j_2, m_2\rangle$ and the properties of the raising and lowering operators $J_\pm = j_{1\pm} + j_{2\pm}$, establish the recurrence relations :

$$\sqrt{J(J+1) - M(M-1)}\, \langle j_1, m_1, j_2, m_2|J, M-1\rangle = \sqrt{j_1(j_1+1) - m_1(m_1+1)}\, \langle j_1, m_1+1, j_2, m_2|J, M\rangle$$
$$\sqrt{j_2(j_2+1) - m_2(m_2+1)}\, \langle j_1, m_1, j_2, m_2+1|J, M\rangle$$
$$\sqrt{J(J+1) - M(M+1)}\, \langle j_1, m_1, j_2, m_2|J, M+1\rangle = \sqrt{j_1(j_1+1) - m_1(m_1-1)}\, \langle j_1, m_1-1, j_2, m_2|J, M\rangle$$
$$\sqrt{j_2(j_2+1) - m_2(m_2-1)}\, \langle j_1, m_1, j_2, m_2-1|J, M\rangle$$

The idea of the Clebsch-Gordan coefficient is to make the change of basis,

$$|L, M\rangle \longrightarrow |l_1, m_1, l_2, m_2\rangle$$

$$|L, M\rangle = \sum_{m_1,m_2} \underbrace{\langle l_1, m_1, l_2, m_2|L, M\rangle}_{\text{Clebsch Gordan Coefficient}} |l_1, m_1, l_2, m_2\rangle$$

$$J_-\, |J, M\rangle = \hbar\sqrt{J(J+1) - M(M-1)}\, |J, M-1\rangle$$

But, we can now use $J_- = (j_{1-} + j_{2-})$,

$$(j_{1-} + j_{2-}) \sum_{m_1,m_2} \langle j_1, m_1, j_2, m_2|J, M\rangle\, |j_1, m_1, j_2, m_2\rangle = \hbar \sum_{m_1,m_2} \langle j_1, m_1, j_2, m_2|J, M\rangle \times$$
$$\Big(\sqrt{j_1(j_1+1) - m_1(m_1-1)}\, |j_1, m_1-1, j_2, m_2\rangle$$
$$+ \sqrt{j_2(j_2+1) - m_2(m_2-1)}\, |j_1, m_1, j_2, m_2-1\rangle\Big)$$

Now, we pre-multiply by $\langle j_1, m_1, j_2, m_2|$,

$$\sum_{m_1',m_2'} \langle j_1, m_1', j_2, m_2'|J, M\rangle \times \Big(\sqrt{j_1(j_1+1) - m_1'(m_1'-1)}\, \langle j_1, m_1, j_2, m_2|j_1, m_1'-1, j_2, m_2'\rangle$$
$$+ \sqrt{j_2(j_2+1) - m_2'(m_2'-1)}\, \langle j_1, m_1, j_2, m_2|j_1, m_1', j_2, m_2'-1\rangle\Big)$$

now, we're using the orthogonality of the basis, so we get for the first term that it is non zero if $m_1' = m_1 + 1$ and $m_2' = m_2$. For the second term, $m_1' = m_1$ and $m_2' = m_2 + 1$. So out of the sum $\sum_{m_1,m_2'}$ there is only two non-zero terms. If we equal the relation on the uncoupled basis and on the coupled basis,

$$\sqrt{J(J+1) - M(M-1)}\, \langle j_1, m_1, j_2, m_1|J, M-1\rangle = \sqrt{j_1(j_1+1) - (m_1+1)m_1}\, \langle j_1, m_1+1, j_2, m_2|J, M\rangle$$
$$+ \sqrt{j_2(j_2+1) - (m_2+1)m_2}\, \langle j_1, m_1, j_2, m_2+1|J, M\rangle$$

2. Using the relations above, show that all Clebsch-Gordan coefficients are real by choosing $\langle j_1, j_1, j_2, J - J_1 | J, J \rangle$ to be real in each fixed total momentum space $J$.

   We specify one Clebsch-Gordan coefficient to be real, *i.e.*, $\langle j_1, j_1, j_2, -J + j_1 | J, J \rangle \in \mathbb{R}$. This is the state with maximum angular momentum projection for both particle $m_1 = j_1$ and $m_2 = -J + j_1$.
   We then use the second relation,

$$\sqrt{J(J+1) - M(M+1)}\, \langle j_1, m_1, j_2, m_1 | J, M+1 \rangle = \sqrt{j_1(j_1+1) - (m_1+1)m_1}\, \langle j_1, m_1 - 1, j_2, m_2 | J, M \\ + \sqrt{j_2(j_2+1) - (m_2+1)m_2}\, \langle j_1, m_1, j_2, m_2 - 1 | J, M \rangle$$

   to show that $\langle j_1, m_1, j_2, m_2 | J, J \rangle \in \mathbb{R}$.

   If we set $M = J$, the left part is 0,

$$\langle j_1, m_1 - 1, j_2, m_2 | J, J \rangle = -\sqrt{\frac{j_2(j_2+1) - m_2(m_2-1)}{j_1(j_1+1) - m_1(m_1-1)}}\, \langle j_1, m_1, j_2, m_2 - 1 | J, J \rangle$$

   if we set $m_1 = j_1$ and $m_2 - 1 = -J + j_1$,

$$\underbrace{\langle j_1, j_1 - 1, j_2, -J - j_1 + 1 | J, J \rangle}_{\in \mathbb{R}} = -\sqrt{\frac{j_2(j_2+1) - m_2(m_2-1)}{j_1(j_1+1) - m_1(m_1-1)}}\, \underbrace{\langle j_1, j_1, j_2, -J + j_1 | J, J \rangle}_{\in \mathbb{R}}$$

   meaning, all the Clebsch-Gordan coefficients in the sub-space $|J, J\rangle$ are real. In order to extend this to all sub-space, we can use the first relation,

$$\langle j_1, m_1, j_2, m_2 | J, J - 1 \rangle = F_1 \langle j_1, m_1 + 1, j_2, m_2 | J, J \rangle + F_2 \langle j_1, m_1, j_2, m_2 + 1 | J, J \rangle$$

   where

$$F_1 = \sqrt{\frac{j_1(j_1+1) - m_1(m_1+1)}{J(J+1) - M(M-1)}} > 0, \qquad F_2 = \sqrt{\frac{j_2(j_2+1) - m_2(m_2+1)}{J(J+1) - M(M-1)}} > 0$$

   meaning, all the Clebsch-Gordan coefficient in the sub-space $|J, J - 1\rangle$ are real. So we can extend this to all sub-spaces.

# Rotations and Wigner Eckart Theorem

Suppose we can define the state of the system in terms of its angular momentum eigenstates and other degrees of freedom. Often we need to compute matrix elements like,

$$\langle k', j', m' | O | k, j, m \rangle$$

If the operator is a tensor operator,

$$O = T_q^{(l)}$$

> **Remark :** $l = 1$ for a three-dimensional vector operator, and $q = \{-1, 0, 1\}$. $q = \pm 1 \rightarrow J_\pm$ and $q = 0 \rightarrow J_x, J_z, J_z$.

then the effect of the operator is to couple angular momentum states in a way as if you had just added new angular momentum to the system.

**Wigner-Eckart Theorem :**

$$\langle j', m' | T_q^{(l)} | j, m \rangle = \underbrace{\langle j', m' | l, q; j, m \rangle}_{C^{lq}_{j',m',j,m}} \frac{\langle j' | | T^{(l)} | | j \rangle}{\sqrt{2j' + 1}}$$

Where $C^{lq}_{j',m',j,m}$ can be recognized as the Clebsch Gordan coefficient.

## Special case, vector operators

### Projection theorem

$$\langle k', j, m' | V_q^{(1)} | k, j, m \rangle = \langle j, m' | J_q | j, m \rangle \underbrace{\frac{\langle k', j, m | \vec{J} \cdot \vec{V} | k, j, m \rangle}{\hbar^2 j(j+1)}}_{\text{scalar}}$$

# EXERCISE IV :

By definition, an observable $\vec{V}$ is vectorial if its three components satisfy the following commutation relations :

$$\begin{aligned} [J_x, V_x] &= 0 \\ [J_x, V_y] &= i\hbar V_z \\ [J_x, V_z] &= -i\hbar V_y \end{aligned}$$

as well as those derived from it by circular permutation of the $x$, $y$ and $z$ indices (e.g., $[J_z, V_x] = i\hbar V_y$, $[J_z, V_y] = -i\hbar V_x$). $\vec{J}$ is the angular momentum with the components $J_x$, $J_y$ and $J_z$.

1. The operators $V_+$, $V_-$, $J_+$ and $J_-$ are defined by

$$V_\pm = V_x \pm iV_y \qquad\qquad J_\pm = J_x \pm iJ_y$$

Show that :

$$\begin{aligned} [J_+, V_+] &= [J_-, V_-] = 0 \\ [J_+, V_-] &= 2\hbar V_z \\ [J_-, V_+] &= -2\hbar V_z \end{aligned}$$

$$\begin{aligned} [J_+, V_\pm] &= [J_x + iJ_y, V_x \pm iV_y] \\ &= [J_x, V_x] \pm i[J_x, V_y] + i[J_y, V_x] \mp [J_y, V_y] \\ &= 0 \pm i(i\hbar V_z) + i(-i\hbar V_z) + 0 \\ &= \mp\hbar V_z + \hbar V_z \end{aligned}$$

Then,

$$[J_+, V_+] = 0 \qquad\qquad [J_+, V_-] = 2\hbar V_z$$

$$\begin{aligned} [J_-, V_\pm] &= [J_x - iJ_y, V_x \pm iV_y] \\ &= [J_x, V_x] \pm i[J_x, V_y] - i[J_y, V_x] \pm [J_y, V_y] \\ &= 0 \pm i[i\hbar V_z] - i(-i\hbar V_z) + 0 \\ &= \hbar V_z(\mp 1 - 1) \end{aligned}$$

And so,

$$[J_-, V_- = 0] \qquad\qquad [J_-, V_+] = -2\hbar V_z$$

2. Establish the following selection rules :

$$\begin{aligned}
\langle k, j, m| V_z |k', j', m'\rangle &= 0 && \text{if } m \neq m' \\
\langle k, j, m| V_+ |k', j', m'\rangle &= 0 && \text{if } m \neq m' + 1 \\
\langle k, j, m| V_- |k', j', m'\rangle &= 0 && \text{if } m \neq m - 1
\end{aligned}$$

where $\{|k, j, m\rangle\}$ is the basis of states in which these operators act.

We start from the commutator $[J_z, V_z]$ to make appear $V_z$,

$$[J_z, V_z] = 0$$

$$\begin{aligned}
\langle k, j, m| [J_z, V_z] |k', j', m'\rangle &= \langle k, j, m| J_z V_z - V_z J_z |k', j', m'\rangle \\
&= (m\hbar - \hbar m') \langle k, j, m| V_z |k' j' m'\rangle = 0
\end{aligned}$$

And so, if $m \neq m'$,

$$\langle k, j, m| V_z |k', j', m'\rangle = 0 \text{ if } m \neq m'$$

In the same way as before, we want to make appear $V_\pm$, so we're going to take the commutator of this observable with one that we know the operation,

$$\begin{aligned}
[J_z, V_\pm] &= [J_z, V_x \pm iV_y] \\
&= [J_z, V_x] \pm i[J_z, V_y] \\
&= i\hbar V_y \pm i(-i\hbar)V_x \\
&= i\hbar V_y \pm \hbar V_x \\
&= \pm\hbar V_\pm
\end{aligned}$$

So,

$$\begin{aligned}
\pm\hbar \langle kjm| V_\pm |k' j' m'\rangle &= \langle kjm| [J_z, V_\pm] |k' j' m'\rangle \\
&= \langle kjm| J_z V_\pm - V_\pm J_z |k' j' m'\rangle \\
&= (m\hbar - m'\hbar) \langle kjm| V_\pm |k' j' m'\rangle
\end{aligned}$$

Thus giving us,

$$\hbar(m - m') \langle k, j, m| V_\pm |k', j', m'\rangle = \pm\hbar \langle k, j, m| V_\pm |k', j', m'\rangle$$

So,

$$\hbar(m - m' \mp 1) \langle k, j, m| V_\pm |k', j', m'\rangle = 0$$

And in the end,

$$\langle k, j, m| V_+ |k', j', m'\rangle = 0 \text{ if } m \neq m' + 1 \qquad \langle k, j, m| V_- |k', j', m\rangle = 0 \text{ if } m \neq m' - 1$$

3. Show that

$$\langle k,j,m+1|\, V_x \,|k,j,m\rangle = \alpha_+(k,j)\,\langle k,j,m+1|\, J_+ \,|k,j,m\rangle$$

where $\alpha_+(k,j)$ is independant of $m$. Then deduce the relation

$$\langle k,j,m|\, V_\pm \,|k,j,m'\rangle = \alpha_\pm(k,j)\,\langle k,j,m|\, J_\pm \,|k,j,m'\rangle$$

where $\alpha_-(k,j)$ is also independent of $m$.

$$[J_+, V_+] = 0$$

$$\langle kjm+1|\, J_+ V_+ \,|kjm-1\rangle = \langle klm+1|\, V_+ J_+ \,|kjm-1\rangle$$

We can insert the completness relation in-between $V_+$ and $J_+$,

$$\text{Id} = \sum_{j'm'} |kj'm'\rangle\,\langle kj'm'|$$

$$\langle jkm+1|\, J_+ \text{ Id } V_+ \,|kjm-1\rangle = \langle klm+1|\, V_+ \text{ Id } J_+ \,|kjm-1\rangle$$

$$\sum_{j'm'} \langle kjm+1|\, J_+ \,|kj'm'\rangle\,\langle kj'm'|\, V_+ \,|kjm-1\rangle = \sum_{j'm'} \langle kjm+1|\, V_+ \,|kj'm'\rangle\,\langle kj'm'|\, J_+ \,|kjm-1\rangle$$

We also know that,

$$\langle jm|j'm'\rangle = \delta_{mm'}\delta_{jj'}$$

So, the sum can be replaced,

$$\langle kjm+1|\, J_+ \,|kjm\rangle\,\langle kjm|\, V_+ \,|kjm-1\rangle = \langle kjm+1|\, V_+ \,|kjm\rangle\,\langle kjm|\, J_+ \,|kjm-1\rangle$$

$$\langle kjm+1|\, V_+ \,|kjm\rangle = \underbrace{\frac{\langle kjm|\, V_+ \,|kjm-1\rangle}{\langle kjm|\, J_+ \,|kjm-1\rangle}}_{\alpha_+(k,j)}\,\langle kjm+1|\, J_+ \,|kjm\rangle$$

this expression hold for,

$$\langle kjm|\, V_+ \,|kjm-1\rangle = \frac{\langle kjm-1|\, V_+ \,|kjm-2\rangle}{\langle kjm-1|\, J_+ \,|kjm-2\rangle}\,\langle kjm|\, J_+ \,|kjm-1\rangle$$

Meaning, this is a recursive expression for all $m$, the only way it can hold is for $\alpha + (k, j)$ independent of $m$.

$$[J_-, V_-] = 0$$

$$\langle kjm+1| J_- V_- |kjm-1\rangle = \langle klm+1| V_- J_- |kjm-1\rangle$$

$$\langle kjm+1| J_- \text{ Id } V_- |kjm-1\rangle = \langle klm+1| V_- - \text{ Id } J_- |kjm-1\rangle$$

$$\langle kjm+1| V_- |kjm\rangle = \underbrace{\frac{\langle kjm| V_- |kjm-1\rangle}{\langle kjm| J_- |kjm-1\rangle}}_{\alpha_-(k,j)} \langle kjm+1| J_- |kjm\rangle$$

4. Show that

$$\langle k,j,m| V_z |k,j,m'\rangle = \alpha(k,j) \langle k,j,m| J_z |k,j,m'\rangle$$

where $\alpha_-(k,j) = \alpha_+(k,j) = \alpha(k,j)$.

As we did earlier, we want to make appear $V_z$, so we take the peculiar commutator $[J_+, V_-]$ that is equal to $2\hbar V_z$ thus making appear $V_z$,

$$[J_+, V_-] = 2\hbar V_z$$

$$2\hbar \langle kjm| V_z |kjm\rangle = \langle kjm| J_+ V_- - V_- J_+ |kjm\rangle$$

But we got an issue, how do we get the operation of $J_+$ onto the bra $\langle|$?

$$J_- |kjm\rangle \qquad \longrightarrow \qquad \langle kjm| \underbrace{(J_-)^\dagger}_{J_+}$$

So,

$$\begin{aligned}
\langle kjm| J_+ V_- - V_- J_+ |kjm\rangle &= \hbar\sqrt{j(j+1) - m(m-1)} \langle kjm-1| V_- |kjm\rangle \\
&\quad - \hbar\sqrt{j(j+1) - m(m+1)} \langle kjm| V_- |kjm+1\rangle \\
&= \hbar\sqrt{j(j+1) - m(m-1)}\alpha_-(k,j) \langle kjm-1| J_- |kjm\rangle \\
&\quad - \hbar\sqrt{j(j+1) - m(m-1)}\alpha_-(k,j) \langle kjm| J_- |kjm+1\rangle \\
&= \hbar^2 \alpha_-(k,j) [j(j+1) - m(m-1) - j(j+1) + m(m+1)] \\
&= 2m\hbar^2 \alpha_-(k,j) \\
&= 2\hbar \langle kjm| V_z |kjm\rangle
\end{aligned}$$

We can do the same with,

$$[J_-, V_+] = -2\hbar V_z$$

$$-2\hbar \langle kjm| V_z |kjm\rangle = \langle kjm| J_-V_+ - V_+J_- |kjm\rangle$$

But we got an issue, how do we get the operation of $J_-$ onto the bra $\langle|$ ?

$$J_+ |kjm\rangle \qquad \longrightarrow \qquad \langle kjm| \underbrace{(J_+)^\dagger}_{J_-}$$

So,

$$\begin{aligned}
\langle kjm| J_-V_+ - V_+J_- |kjm\rangle &= \hbar\sqrt{j(j+1) - m(m+1)}\, \langle kjm+1| V_+ |kjm\rangle \\
&\quad - \hbar\sqrt{j(j+1) - m(m-1)}\, \langle kjm| \langle kjm| V_+ |kjm-1\rangle \\
&= \hbar\sqrt{j(j+1) - m(m+1)}\alpha_+(k,j) \langle kjm+1| J_+ |kjm\rangle \\
&\quad - \hbar\sqrt{j(j+1) - m(m-1)}\alpha_+(k,j) \langle kjm| \langle kjm| J_+ |kjm-1\rangle \\
&= \hbar^2\alpha_+(k,j) \left[j(j+1) - m(m+1) - j(j+1) + m(m-1)\right] \\
&= -2m\hbar^2\alpha_+(k,j) \\
&= -2\hbar \langle kjm| V_z |kjm\rangle
\end{aligned}$$

Then,

$$\alpha_+(k,j) = \alpha_-(k,j) = \alpha(k,j)$$

And we found the same result, so $\alpha_-$ and $\alpha_+$ are indeed the same coefficient.

5. Deduce that

$$|k,j,m\rangle \vec{V} |k,j,m'\rangle = \alpha(k,j) \langle k,j,m| \vec{J} |k,j,m'\rangle$$

By circular permutation of the indexes, we do the same thing as before, and so it's verified.

6. Show that

$$\alpha(k,j) = \frac{\langle k,j,m| \vec{J}\cdot\vec{V} |k,j,m\rangle}{j(j+1)\hbar^2}$$

$\vec{J}\cdot\vec{V}$ is a compact writing for $J_xV_x + J_yV_y + J_zV_z$.

$$\begin{aligned}
\langle kjm| J_xV_x |kjm\rangle &= \sum_{k_1j_1m_1} \langle kjm| J_x |k_1j_1m_1\rangle \langle k_1j_1m_1| V_x |kjm\rangle \\
&= \sum_{m_1} \langle kjm| J_x |kjm_1\rangle \langle kjm_1| V_x |kjm\rangle \\
&= \alpha(k,j) \sum_{m_1} \langle kjm| J_x |kjm_1\rangle \langle kjm_1| J_x |kjm\rangle \\
&= \alpha(k,j) \langle kjm| J_x^2 |kjm\rangle \\
&= \alpha(k,j) \left(j(j+1)\hbar^2\right)
\end{aligned}$$

So, we can perform a circular permutation of the indexes and we find immediately,

$$\langle k, j, m | \vec{J} \cdot \vec{V} | k, j, m \rangle = \alpha(k, j) \langle kjm | J^2 | kjm \rangle$$

So,

$$\alpha(k, j) = \frac{\langle kjm | \vec{J} \cdot \vec{V} | kjm \rangle}{j(j+1)\hbar^2}$$