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Radiation Matter Interaction
Time-dependent perturbation theory

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A) Lecture revision and interpretation

Within time-dependent perturbation theory exposed in the lecture, the transition probability between two state i and k is given by

$$P_{ik} = \frac{1}{\hbar^2}\left|\int_{t_0}^{t} \mathrm{d}t' W(t') \exp\left(\frac{i(E_k - E_i)t'}{\hbar}\right)\right|^2$$

1. For the case of a two-level system, revisit the lecture material and justify the expression for the transition probability, between the two discrete states i and f , under the action of a sinusoidal perturbation:

$$P_{if} = \frac{|W_{fi}|^2}{4\hbar^2}t^2\left(\frac{\sin\left(\frac{(\omega_{fi}-\omega)}{2}t\right)}{\frac{\omega_{fi}-\omega}{2}t}\right)^2$$

   with $\omega_{fi} = (E_f - E_i)/\hbar$. Plot the result as a function of ω, and discuss the time range of validity, keeping in mind that $P_{ik} \leq 1$.

2. For a constant perturbation, derive the result found in the lecture:

$$P_{if} = \frac{|W_{fi}|^2}{\hbar^2}t^2\left(\frac{\sin\left(\frac{\omega_{fi}t}{2}\right)}{\left(\frac{\omega_{fi}t}{2}\right)}\right)^2$$

3. Trace this probability for different values of t, and give an intuitive description of the time-dependent transition probability.

B) Two-level system with a constant non-resonant perturbation, full treatment: Rabi oscillations

We explore here an alternative, more general treatment of perturbations, which, in the case of a two-level system, is simpler in formalism and can be used without limitations to the strength/amplitude of the perturbation interaction W. In this context, we'll introduce the two limits of "strong" and "weak" coupling, which are frequently used concepts for the description of light-matter interactions of QM systems (e.g. atoms/molecules in an EM cavity).

1. We study a two-level system, defined by the orthogonal eigenstates of an unpertubed Hamiltonian H_0.

   $$H_0 \left|\varphi_1\right\rangle = E_1 \left|\varphi_1\right\rangle \qquad H_0 \left|\varphi_2\right\rangle = E_2 \left|\varphi_2\right\rangle \qquad \left\langle\varphi_i|\varphi_j\right\rangle = \delta_{ij}$$

   Under the effect of a perturbation W, the full Hamiltonian is $H = H_0 + W$. In the unperturbed basis W is a Hermitian matrix:

   $$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

   with $W_{21} = W_{12}^*$. The system is described by new eigenstates and eigenenergies E_+ and E_-:

   $$H \left|\varphi_+\right\rangle = E_+ \left|\varphi_+\right\rangle \qquad H \left|\varphi_-\right\rangle = E_- \left|\varphi_-\right\rangle$$

   Show that the new eigenenergies are given by

   $$E_\pm = \frac{1}{2}(E_1 + W_{11} + E_2 + W_{22}) \pm \frac{1}{2}\sqrt{(E_1 + W_{11} - E_2 - W_{22})^2 + 4|W_{12}|^2}$$

   It can be shown (cf. Cohen-Tannoudji, vol. 1, supplement $\mathrm{B_{IV}}$) that the eigenstates are given by

   $$\begin{cases} \left|\psi_+\right\rangle = \cos\left(\dfrac{\theta}{2}\right) e^{-i\varphi/2} \left|\varphi_1\right\rangle + \sin\left(\dfrac{\theta}{2}\right) e^{+i\varphi/2} \left|\varphi_2\right\rangle \\ \left|\psi_-\right\rangle = -\sin\left(\dfrac{\theta}{2}\right) e^{-i\varphi/2} \left|\varphi_1\right\rangle + \cos\left(\dfrac{\theta}{2}\right) e^{+i\varphi/2} \left|\varphi_2\right\rangle \end{cases}$$

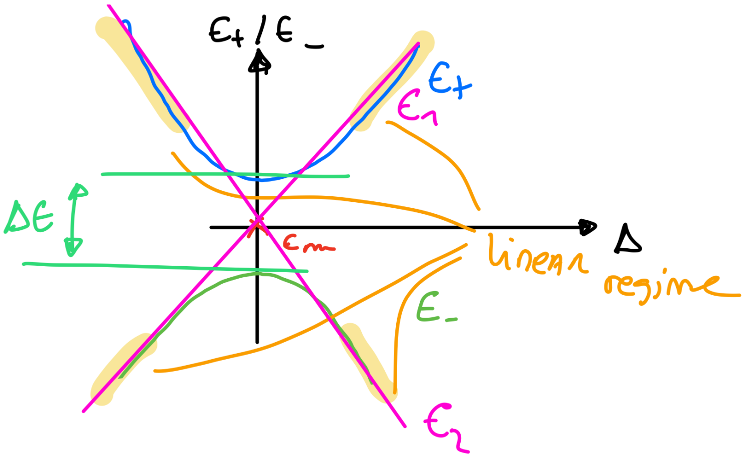
   with

   $$\tan\theta = \frac{2|W_{12}|}{E_1 + W_{11} - E_2 - W_{22}} \qquad W_{21} = |W_{12}| e^{i/varphi}$$

   Cohen-Tannoudji T1- B4

2. A graphical representation of the eigenenergies. The perturbation mixes states 1 and 2 due to the non-diagonal matrix elements W_{12}. We will only retain these in the following and assume $W_{11} = W_{22} = 0$. Introduce the quantities $E_m = \frac{1}{2}(E_1 + E_2)$, and $\Delta = \frac{1}{2}(E_1 - E_2)$, and draw the two branches of $E_\pm$, as a function of Δ. When the energy axis of $/Delta$ crosses the energy axis in the ordinate E_m, the curves for $E_\pm$, are two hyperbolas which are symmetric to the coordinate axes. E_1 and E_2 can be represented as straight lines with slope $+1$ and -1, respectively.

- Describe how the effect of the perturbation $E_+ - E_1$ ($\Delta > 0$) or $E_+ - E_2$ ($\Delta < 0$) changes as a function of $|\Delta|$.
- What is the value of the "resonance splitting" for $\Delta = 0$?
- Give examples of quantum mechanical systems, which show modified eigenergies due to such non-diagonal interaction terms (W_{12}).



For the resonance splitting, *i.e.* $\Delta = 0$, we get $\Delta E = E_+ - E_- = 2|W_{12}|$. For large Δ, there is a *small* effect of the perturbation. The correction to the eigenenergies is the strongest for resonance interaction.

Examples of systems that showcase this would be,

- Optical lattices
- Quantum dots
- Quantum well
- Covalent bonding

3. The dynamic evolution. The state of the whole system is in a superposition of the eigenstates

$$|\Psi(t)\rangle \, \lambda e^{-iE_+t/\hbar} |\Psi_+\rangle + \mu e^{-iE_-t/\hbar} |\Psi_-\rangle$$

In order to obtain the time-dependent evolution in the states ϕ_1 and ϕ_2, we'll assume that the system is initially in $|\Psi(t=0)\rangle = |\varphi_1\rangle$ and project its evolution on the basis ϕ_1 and ϕ_2.

a) Express $|\Psi(t=0)\rangle = |\varphi_1\rangle$ in the basis $|\Psi_+\rangle$ and $|\Psi_-\rangle$.

b) Write the time-dependence of $|\Psi(t)\rangle$ in this basis.

c) Show that the time-dependent probability for transition into $|\varphi_2\rangle$ is given by

$$P_{12}(t) = \frac{4|W_{12}|^2}{4|W_{12}|^2 + (E_1 - E_2)^2} \sin^2\left(\sqrt{4|W_{12}|^2 + (E_1 - E_2)^2}\frac{t}{2\hbar}\right)$$

This is called the "Rabi transition probability".

$$|\psi(t=0)\rangle = \lambda\,|\psi_+\rangle + \mu\,|\psi_-\rangle = |\varphi_1\rangle$$

Thus,

$$\begin{aligned}|\varphi_1\rangle &= \lambda\left[\cos\left(\frac{\theta}{2}\right)e^{-i\varphi/2}\,|\varphi_1\rangle + \sin\left(\frac{\theta}{2}\right)e^{+i\varphi/2}\,|\varphi_2\rangle\right]\\ &\quad + \mu\left[-\sin\left(\frac{\theta}{2}\right)e^{-i\varphi/2}\,|\varphi_1\rangle + \cos\left(\frac{\theta}{2}\right)e^{+i\varphi/2}\,|\varphi_2\rangle\right]\\ &= |\varphi_1\rangle\,e^{-i\varphi/2}\left[\lambda\cos\left(\frac{\theta}{2}\right) - \mu\sin\left(\frac{\theta}{2}\right)\right] + |\varphi_2\rangle\,e^{+i\varphi/2}\left[\lambda\sin\left(\frac{\theta}{2}\right) + \mu\cos\left(\frac{\theta}{2}\right)\right]\end{aligned}$$

Leading us to,

$$\lambda\sin\left(\frac{\theta}{2}\right) + \mu\cos\left(\frac{\theta}{2}\right) = 0$$

Making a link in between λ and μ,

$$\mu = -\lambda\tan\left(\frac{\theta}{2}\right)$$

Plugging it in back, we get

$$\lambda = \cos\left(\frac{\theta}{2}\right)e^{i\varphi/2}$$

Meaning,

$$\begin{cases}\lambda = e^{+i\varphi/2}\cos\left(\frac{\theta}{2}\right)\\ \mu = -e^{+i\varphi/2}\sin\left(\frac{\theta}{2}\right)\end{cases}$$

Meaning that we can get,

$$|\psi(t)\rangle = e^{i\varphi/2}\cos\left(\frac{\theta}{2}\right)e^{-iE_+t/\hbar}\,|\psi_+\rangle - e^{i\varphi/2}\sin\left(\frac{\theta}{2}\right)e^{-iE_-t/\hbar}\,|\psi_-\rangle$$

And we know that,

$$P_{12}(t) = |\,\langle\varphi_2|\psi(t)\rangle\,|^2$$

going to the end of this calculation indeed allows us to get the correct result.