Exam — Session 1

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1 The Blume–Capel model [14 points]

The Blume–Capel model describes a magnetic material with some nonmagnetic vacancies. Let us consider a lattice [we denote by \( N \gg 1 \) the number of lattice sites and by \( z \) the number of nearest neighbors] of spins \( S_i \) that can take the values \(-1, 0, +1\). A spin 0 corresponds to a vacancy (nonmagnetic impurity or empty site) and spins +1 or −1 correspond to the two different orientations of the magnetic species. We assume that the Hamiltonian of the system in presence of an homogeneous magnetic field \( h \) is given by

\[
H = -J \sum_{\langle i,j \rangle} S_i S_j + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \tag{1.1}
\]

where \( J > 0 \) is the exchange interaction and where \( \Delta \) is a constant that can be either negative or positive. In the Hamiltonian above, \( \langle i, j \rangle \) denotes a summation over nearest neighbors.

1.1 General discussion

(a) Justify that \(-\Delta\) is the energy of creation of a vacancy. In which case (\( \Delta > 0 \) or \( \Delta < 0 \)) is it favorable to create a vacancy?

It is favorable to create a vacancy when there is a diminution of energy when one is created, meaning that \(+\Delta\) is negative.

(b) At \( T = 0 \) and \( h = 0 \), calculate the energy of the system in the three different states \( \langle S_i \rangle = +1, \langle S_i \rangle = -1, \) and \( \langle S_i \rangle = 0 \). Which state is selected at \( T = 0 \)?

\[
\langle H \rangle = -J \sum_{\langle i,j \rangle} \langle S_i S_j \rangle + \Delta \sum_{i=1}^{N} \langle S_i^2 \rangle - h \sum_{i=1}^{N} \langle S_i \rangle
\]

\[
= -J \sum_{\langle i,j \rangle} \langle S_i S_j \rangle + \Delta \sum_{i=1}^{N} \langle S_i^2 \rangle
\]

\[
C_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle
\]

In a mean-field-approximation we neglect correlations, meaning,

\[
C_{ij} = 0 \implies \langle S_i S_j \rangle = \langle S_i \rangle \langle S_j \rangle
\]

\[
\langle H \rangle = -J \sum_{\langle i,j \rangle} \langle S_i \rangle \langle S_j \rangle + \Delta \sum_{i=1}^{N} \langle S_i \rangle \langle S_i \rangle
\]
\[ \langle H \rangle_{+1} = -J \sum_{\langle i,j \rangle} 1 + \Delta \sum_{i=1}^{N} 1 = -\frac{1}{2} J N z + \Delta N = N \left( \Delta - \frac{J z}{2} \right) \]

\[ \langle H \rangle_{-1} = -J \sum_{\langle i,j \rangle} 1 + \Delta \sum_{i=1}^{N} 1 = -\frac{1}{2} J N z + \Delta N = N \left( \Delta - \frac{J z}{2} \right) \]

\[ \langle H \rangle_{0} = -J \sum_{\langle i,j \rangle} 0 + \Delta \sum_{i=1}^{N} 0 = 0 \]

The selected state is the one with the lowest energy, meaning that, if \( \Delta - \frac{J z}{2} < 0 \) it is the state \( \langle S_i \rangle = \pm 1 \) that is selected, and if \( \Delta - \frac{J z}{2} > 0 \) it is the state \( \langle S_i \rangle = 0 \) that is selected.

(c) Which limit of \( \Delta \) corresponds to the usual two-state Ising model? How would you call the \( \Delta = 0 \) model?

For \( \Delta = 0 \) we recover the usual Ising model.

1.2 Mean-field approximation

We now aim at performing a mean-field approximation (MFA). We write \( S_i = m + \delta S_i \), where \( m = \langle S_i \rangle \) is the average magnetization.

(a) Define the spin-spin correlation function \( C_{ij} \). What is the value of \( C_{ij} \) in the MFA?

\[
C_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \\
= \langle (m_i + \delta S_i)(m_j + \delta S_j) \rangle - m_i m_j \\
= \langle \delta S_i \delta S_j \rangle
\]

Within the MFA, we’re neglecting the correlation between spin-spin, meaning,

\[ C_{ij} = 0 \]

(b) Show that within the MFA, it is possible to write the Hamiltonian (1.1) as

\[
\mathcal{H} \simeq \frac{1}{2} N z J m^2 - (h + z J m) \sum_{i=1}^{N} S_i + \Delta \sum_{i=1}^{N} S_i^2.
\]
\[ H = -J \sum_{\langle i,j \rangle} S_i S_j + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = -J \sum_{\langle i,j \rangle} (m_i + \delta S_i)(m_j + \delta S_j) + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = -J \sum_{\langle i,j \rangle} (m_i m_j + m_i \delta S_j + m_j \delta S_i + \delta S_i \delta S_j) + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ \approx -J \sum_{\langle i,j \rangle} m^2 - 2J \sum_{\langle i,j \rangle} m \delta S_i + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = -\frac{1}{2} N z J m^2 - 2Jm \sum_{\langle i,j \rangle} (S_i - m) + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = -\frac{1}{2} N z J m^2 - 2Jm \sum_{\langle i,j \rangle} S_i + 2Jm^2 \sum_{\langle i,j \rangle} (1) + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = \frac{1}{2} N z J m^2 - 2Jm \sum_{\langle i,j \rangle} S_i + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = \frac{1}{2} N z J m^2 - z Jm \sum_{i=1}^{N} S_i + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \]

\[ = \frac{1}{2} N z J m^2 - (h + z Jm) \sum_{i=1}^{N} S_i + \Delta \sum_{i=1}^{N} S_i^2 \]

\[ h - z Jm \] can be seen as an effective field \( h_{\text{eff}} \).

(c) Calculate the free energy \( F \) within the MFA.
\[ Z = \sum_{S_1} \cdots \sum_{S_N} e^{-\beta H} \]
\[ = \sum_{S_1} \cdots \sum_{S_N} e^{-\beta \left[ \frac{1}{2} N z J m^2 - (h + z J m) \sum_{i=1}^N S_i + \Delta \sum_{i=1}^N S_i^2 \right]} \]
\[ = e^{-\frac{1}{2} \beta N z J m^2} \prod_{\{S_i\}} \sum_{S_i=+1,0,-1} e^{\beta(h + z J m)S_i - \beta \Delta S_i^2} \]
\[ = e^{-\frac{1}{2} \beta N z J m^2} \prod_{i} \left[ 1 + e^{\beta(h + z J m) - \beta \Delta} + e^{-\beta(h + z J m) - \beta \Delta} \right] \]
\[ = e^{-\frac{1}{2} \beta N z J m^2} \prod_{i} \left[ 1 + 2e^{-\beta \Delta} \cosh(\beta(h + z J m)) \right] \]
\[ = e^{-\frac{1}{2} \beta N z J m^2} \left[ 1 + 2e^{-\beta \Delta} \cosh(\beta(h + z J m)) \right]^N \]

which makes it possible to find an expression for free energy. Indeed, we have
\[ F = -k_B T \ln Z \]
\[ = \frac{N z J m^2}{2} - N k_B T \ln \left\{ 1 + 2e^{-\beta \Delta} \cosh(\beta(h + z J m)) \right\} \]

(d) Demonstrate that the average value \( m = \langle S_i \rangle \) is given by the expression
\[ m = \frac{1}{N} \frac{\partial F}{\partial h} \]

Deduce that, within the MFA, the magnetization obeys the self-consistent equation (SCE)
\[ m = \frac{2 \sinh(\beta[h + z J m])}{\exp(\beta \Delta) + 2 \cosh(\beta[h + z J m])}. \]

\[ m_i = \langle S_i \rangle \]
\[ = \sum_{S_1} \cdots \sum_{S_N} S_i \sum_{S_i=\pm 1} e^{\beta h} \]
\[ = 1 \frac{\partial \ln \Xi}{\beta \partial h_i} \]
\[ = \frac{\partial F}{\partial h_i} \]
Since we take the mean value of $m$’s,

$$m = \frac{\sum m_i}{N} = -\frac{1}{N} \frac{\partial F}{\partial h}$$

$$m = -\frac{1}{N} \frac{\partial F}{\partial h}$$

$$= -\frac{1}{N} \frac{\partial}{\partial h} \left\{ \frac{N z J m^2}{2} - N k_B T \ln \left\{ 1 + 2 e^{-\beta \Delta} \cosh(\beta (h + z J m)) \right\} \right\}$$

$$= k_B T \frac{\partial}{\partial h} \left\{ \ln \left\{ 1 + 2 e^{-\beta \Delta} \cosh(\beta (h + z J m)) \right\} \right\}$$

$$= k_B T \frac{2 e^{-\beta \Delta} \beta \sinh(\beta (h + z J m))}{1 + 2 e^{-\beta \Delta} \cosh(\beta (h + z J m))}$$

$$= \frac{2 \sinh(\beta [h + z J m])}{\exp(\beta \Delta) + 2 \cosh(\beta [h + z J m])}$$

From now on, we consider the case of vanishing magnetic field, $h = 0$.

(e) In the case $\Delta \to -\infty$, discuss the solutions of the SCE.

For $\Delta \to -\infty$,

$$m = \tanh[\beta z J m]$$

We can define a critical temperature,

$$k_B T_c = z J$$

If $T > T_c$, there is only one solution : $m = 0$, if $T < T_c$, there are three solutions : $m = 0$, $m = \pm m(T)$. Ultimately, if $T = T_c$, there is only one solution : $m = 0$.

(f) In the general case, show that $m = 0$ is a solution of the SCE.

$$m = \frac{2 \sinh(\beta [z J m])}{\exp(\beta \Delta) + 2 \cosh(\beta [z J m])}$$

We know that $\sinh(0) = 0$ and $\cosh(0) = 1$, so, $m = 0$ is always a solution because :

$$0 = \frac{0}{e^{\beta \Delta} + 2}$$

Hence, it is always a solution.
(g) We now aim at discussing graphically the solutions of the SCE. We define \( t = k_B T / zJ \) and \( \delta = \Delta / zJ \).

(i) Express the SCE in term of the function

\[
f(m) = \frac{2 \sinh(m/t)}{\exp(\delta/t) + 2 \cosh(m/t)}.
\]

\[
m = \frac{2 \sinh(\beta[zJm])}{\exp(\beta \Delta) + 2 \cosh(\beta[zJm])} = \frac{2 \sinh(zJm/k_B T)}{\exp(\Delta/k_B T) + 2 \cosh(zJm/k_B T)}
\]

\[
\delta = \frac{\Delta}{k_B T}.
\]

(ii) What is the value of \( f(0) \)?

\[
f(0) = 0
\]

(iii) What are the limits of \( f(m) \) when \( m \to \pm \infty \)?

\[
\lim_{m \to -\infty} f(m) = \lim_{m \to -\infty} \frac{e^{m/t} - e^{-m/t}}{e^{\delta/t} + e^{\delta/t} + e^{-m/t}} = \lim_{m \to -\infty} \frac{-e^{-m/t - \delta/t}}{1 + e^{-m/t - \delta/t}} \approx -1.
\]

\[
\lim_{m \to +\infty} f(m) = \lim_{m \to +\infty} \frac{e^{m/t} - e^{m/t}}{e^{\delta/t} + e^{\delta/t} + e^{-m/t}} = \lim_{m \to +\infty} \frac{e^{m/t - \delta/t}}{1 + e^{m/t - \delta/t}} \approx 1.
\]

(iv) Calculate

\[
\left. \frac{df}{dm} \right|_{m=0}
\]
and discuss graphically the number of solutions to the SCE. Show that there is a critical reduced temperature $t_c$ defined by the equation

$$t_c = \frac{2}{2 + \exp(\delta/t_c)}.$$

$$\frac{df}{dm} f(m) = \frac{d}{dm} \left( \frac{2 \sinh(m/t)}{\exp(\delta/t) + 2 \cosh(m/t)} \right) \bigg|_{m=0}$$

$$= \frac{\frac{2}{t} \cosh(m/t)(\exp(\delta/t) + 2 \cosh(m/t)) - 2 \sinh(m/t) \left( \frac{2}{t} \sinh(m/t) \right)}{(\exp(\delta/t) + 2 \cosh(m/t))^2} \bigg|_{m=0}$$

$$= \frac{\frac{2}{t}(\exp(\delta/t) + 2)}{(\exp(\delta/t) + 2)^2}$$

$$= \frac{2/t}{\exp(\delta/t) + 2}$$

But the left-side is also equal to 1, so

$$1 = \frac{2/t}{\exp(\delta/t) + 2} \iff t = \frac{2}{2 + \exp(\delta/t)}$$

(v) In figure 1 (colored lines) is plotted the function $g(t, \delta) = 2/[2 + \exp(\delta/t)]$ as a function of $t$ for different values $\delta_i$ of $\delta$. Which $\delta_i$'s are positive and which of them are negative? Sort by ascending order the $\delta_i$'s.
\[ g(t, \delta) = \frac{2}{2 + \exp(\delta/t)} \]

if $\delta$ is positive, when $t \rightarrow 0$, the exponential goes to $\infty$ meaning that $g(t, \delta) = 0$. If $\delta$ is negative, when $t \rightarrow 0$, the exponential goes to $0$ meaning that $g(t, \delta) = 2/2 = 1$. So looking at the curves, the red one ($\delta_4$) is for a negative $\delta$ and the three others are positive.

(vi) Plot the curve $g(t, \delta)$ for the value of $\delta$ corresponding to the Ising model and give the corresponding $t_c$.

For the Ising model,

\[ \delta = \frac{\Delta}{zJ} = 0 \]

So,

\[ g(t, \delta) = 1 \]

Meaning $t_c = 1$.

(vii) Using your previous discussion and question 1.1(b), sketch the general behavior of $t_c$ as a function of $\delta$. 
2 Pauli paramagnetism of a two-dimensional electron gas [6 points]

In this exercise, we wish to understand one of the magnetic properties of a noninteracting electron gas: the Pauli paramagnetism which is due to the alignment of the electronic magnetic moments with the applied magnetic field. The one-electron Hamiltonian describing this phenomenon is given by

\[ H = \frac{p^2}{2m} - \mu - zB. \]  

(2.1)

Here, \( p \) is the electron momentum, \( m \) its mass, \( \mu_z = qS_z/m \) its magnetic moment, which is related to its spin \( S_z \) through the gyromagnetic factor \( \gamma = q/m \), where \( q = -e \) (with \( e = 1.6 \times 10^{-19} \text{ C} \) the elementary charge). In what follows, we consider a homogeneous magnetic field \( B \) parallel to the \( z \) axis, and we assume that electrons are confined to a two-dimensional rectangular surface with area \( A = L_xL_y \), where \( L_x \) and \( L_y \) are the lateral dimensions of the electron gas in the \( x \) and \( y \) directions, respectively.

We recall that electrons are spin 1/2 particles, so that they obey the Fermi–Dirac statistics. The average occupancy of an energy state \( \epsilon \) is then given by

\[ f(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \]  

(2.2)

where \( \beta = 1/k_B T \), with \( T \) the temperature of the gas, and where \( \mu = \mu(T) \) is the chemical potential.

2.1 Warm up

(a) Plot the Fermi–Dirac distribution (2.2) for both \( T = 0 \) and \( T \neq 0 \).
(b) How is defined the Fermi energy $\epsilon_F$ in terms of $\mu$?

$$\epsilon_F = \mu(T = 0)$$

(c) In absence of magnetic field, the Hamiltonian (2.1) reduces to

$$H = \frac{p^2}{2m} \quad (2.3)$$

Using periodic boundary conditions, one can easily show that the spectrum corresponding to the Hamiltonian (2.3) is given by $\epsilon_k = \hbar^2 k^2 / 2m$, where the wavevector $\mathbf{k} = (k_x, k_y)$ is quantized according to $k_x = 2\pi n_x / L_x$ and $k_y = 2\pi n_y / L_y$, with $n_x$ and $n_y$ integer numbers. Show that the corresponding density of states (including the spin degeneracy) is energy independent and is given in the thermodynamic limit by

$$\rho_0 = \frac{m A}{\pi \hbar^2}. \quad (2.4)$$
\[ \Omega(0 \rightarrow E) = \frac{2\pi |k|^2}{(2\pi)^2} \frac{2mE}{L_x L_y \hbar^2} = \frac{AmE}{\pi\hbar^2} \]

The factor 2 in here is for the spin degeneracy. And so,
\[
\rho(E) = \frac{d\Omega(0 \rightarrow E)}{dE} = \frac{Am}{\pi\hbar^2}
\]

(d) Still in absence of a magnetic field, show that \( \epsilon_F = N/\rho_0 \), where \( N \) is the total number of electrons in the two-dimensional gas.

\[
\epsilon_F = \frac{\hbar^2 k_F^2}{2m} \quad \quad k_F = \sqrt{\frac{2m\epsilon_F}{\hbar^2}}
\]

\[
N = \frac{k_F^2}{(2\pi)^2} \frac{2m\epsilon_F}{L_x L_y} \frac{Am}{\pi\hbar^2} = \epsilon_F \rho_0
\]

And so,
\[
\epsilon_F = \frac{N}{\rho_0}
\]

2.2 Pauli paramagnetism

The energy spectrum corresponding to the Hamiltonian (2.1) is spin dependent, and given by
\[
\epsilon_k^\pm = \frac{\hbar^2 |k|^2}{2m^*} \mp \epsilon_B,
\]

where \( (+) \) corresponds to a spin up (down) electron. Here, \( \epsilon_B = \mu_B B \), with \( \mu_B = \hbar q/2m \) the Bohr magneton.

(a) Show that the density of states of the two spin species is energy dependent and given by
\[
\rho_\pm(\epsilon) = \frac{1}{2} \rho_0 \theta(\epsilon \pm \epsilon_B),
\]

where \( \theta(x) \) is the Heaviside step function.
At $T = 0$,

$$
\epsilon^+ = \epsilon_k - \epsilon_B = \frac{\hbar^2 k^2}{2m} - \epsilon_B
$$

$$
\rho_+(\epsilon) d\epsilon = \frac{1}{2} \frac{A m}{\pi \hbar^2} (\epsilon + \epsilon_B) d\epsilon
$$

And we know that $\rho_+(\epsilon) d\epsilon > 0$, implying that $\epsilon + \epsilon_B > 0$. But, if $\epsilon + \epsilon_B < 0$, this does mean that $\hbar^2 k^2/2m < 0$, which is impossible, meaning there is not states like that and so we get,

$$
\rho_+(\epsilon) = \frac{1}{2} \frac{A m}{\pi \hbar^2} \theta(\epsilon + \epsilon_B)
$$

$$
= \frac{1}{2} \rho_0 \theta(\epsilon + \epsilon_B)
$$

And same for $\rho_-$ so we get,

$$
\rho_\pm(\epsilon) = \frac{1}{2} \rho_0 \theta(\epsilon \pm \epsilon_B)
$$

where the $1/2$ factor is here because of the lift of the degeneracy from the without field case.

(b) Let us first assume that both the temperature $T$ and the chemical potential $\mu$ are fixed. Show that the average number of spin up and spin down electrons, denoted by $N_\pm$, is given by

$$
N_\pm = \frac{\rho_0}{2 \beta} \ln \left( 1 + e^{\beta \frac{\epsilon}{\epsilon_B + \mu}} \right).
$$

$$
N_+ = \int d\epsilon \rho_+(\epsilon) f(\epsilon)
$$

$$
= \int d\epsilon \frac{1}{2} \rho_0 \theta(\epsilon + \epsilon_B) \frac{1}{e^{\beta(\epsilon - \mu)} + 1}
$$

$$
N_+ = \frac{1}{2} \rho_0 \int_{-\epsilon_B}^{+\infty} d\epsilon \frac{1}{e^{\beta(\epsilon - \mu)} + 1}
$$

$$
= \frac{1}{2} \rho_0 \int_{-\epsilon_B}^{+\infty} d\epsilon e^{-\beta(\epsilon - \mu)}
$$

$$
= -\frac{1}{\beta} \rho_0 \int_{-\epsilon_B}^{+\infty} d\epsilon e^{-\beta(\epsilon - \mu)}
$$

$$
= \frac{\rho_0}{2 \beta} \ln \left( 1 + e^{\beta (\epsilon - \mu)} \right)
$$

$$
N_- = \frac{\rho_0}{2 \beta} \ln \left( 1 + e^{\beta (\epsilon + \mu)} \right)
$$
\[ N_- = \int_{-\infty}^{+\infty} d\epsilon \frac{1}{2} \rho_0 \theta(\epsilon - \epsilon_B) \frac{1}{1 + e^{\beta(\epsilon - \mu)}} \]
\[ = \rho_0 \int_{\epsilon_B}^{+\infty} -\beta e^{-\beta(\epsilon - \mu)} \frac{1}{e^{-\beta(\epsilon - \mu)} + 1} \]
\[ = -\frac{\rho_0}{2\beta} \left[ \ln(1 + e^{-\beta(\epsilon - \mu)}) \right]_{\epsilon_B}^{+\infty} \]
\[ = \frac{\rho_0}{2\beta} \ln(1 + e^{-\beta(\epsilon - \mu)}) \]

In the end,
\[ N_\pm = \frac{\rho_0}{2\beta} \ln(1 + e^{\beta(\pm\epsilon_B + \mu)}) \]

(c) Let us now consider that the total number of electrons \( N = N_+ + N_- \) is fixed. Deduce from the preceding question a quadratic equation for the fugacity \( z = e^{\beta\mu} \). Give the resulting expression of the chemical potential as a function of \( \epsilon_F \) and \( \epsilon_B \). In particular, analyze the low temperature \( (\beta\epsilon_F \gg 1) \) and low magnetic field \( (\beta\epsilon_B \ll 1) \) limits.

\[ N_\pm = \frac{\rho_0}{2\beta} \ln(1 + ze^{\pm\beta\epsilon_B}) \]

\[ N = \frac{\rho_0}{2\beta} \ln \left[ (1 + ze^{\beta\epsilon_B}) (1 + ze^{-\beta\epsilon_B}) \right] \]
\[ = \frac{\rho_0}{2\beta} \ln(1 + 2z \cosh(\beta\epsilon_B) + z^2) \]

Meaning,
\[ \frac{2\beta N}{\rho_0} = \ln(1 + 2z \cosh(\beta\epsilon_B) + z^2) \]

\[ z^2 + 2z \cosh(\beta\epsilon_B) + 1 - e^{\frac{2N}{\rho_0}} = 0 \]

The solution of this polynomial is
\[ z = \frac{-2 \cosh(\beta\epsilon_B) + \sqrt{4 \cosh^2(\beta\epsilon_B) - 4(1 - e^{\frac{2N}{\rho_0}})}}{2} > 0 \]
We know that \( z = e^{\beta \mu} \) and so,

\[
\mu = \frac{1}{\beta} \ln \left\{ -2 \cosh(\beta \epsilon_B) + \sqrt{4 \cosh^2(\beta \epsilon_B) - 4(1 - e^{2\beta N_{\rho_0}})} \right\}
\]

\[
\lim_{T \to 0, B \to 0} \mu = \epsilon_F
\]

If \( B = 0, \epsilon_B = 0 \), meaning,

\[
\mu_{B=0} = \frac{1}{\beta} \ln \left\{ e^{\beta N_{\rho_0}} - 1 \right\}
\]

\[
\mu_{B=0, \beta \to \infty} = \frac{1}{\beta} \ln \left( e^{\beta N_{\rho_0}} \right) = \frac{N}{\rho_0}
\]

And thus we recover

\[
\epsilon_F = \frac{N}{\rho_0}
\]

(d) The magnetization of the electron gas is given by \( M = \mu_B(N_+ - N_-)/A \), and the corresponding susceptibility is defined as

\[
\chi_P = \lim_{B \to 0} \frac{\partial M}{\partial B}.
\]

Calculate the Pauli susceptibility \( \chi_P \) as a function of \( \rho_0, A, \) and \( \mu_B \) in the degenerate limit \( \beta \epsilon_F \gg 1 \).

\[
N_+ = \frac{\rho_0}{2\beta} \ln \left( 1 + e^{\beta(\epsilon_B + \mu)} \right) \quad N_- = \frac{\rho_0}{2\beta} \ln \left( 1 + e^{\beta(-\epsilon_B + \mu)} \right)
\]

\[
N_+ - N_- = \frac{\rho_0}{2\beta} \ln \left( \frac{1 + e^{\beta(\epsilon_B + \mu)}}{1 + e^{\beta(-\epsilon_B + \mu)}} \right)
\]

But we are at \( \beta \epsilon_F \gg 1 \),

\[
\mu = \frac{1}{\beta} \ln \left\{ \sqrt{\cosh^2(\beta \epsilon_B) + e^{2\beta \epsilon_F}} - 1 - \cosh(\beta \epsilon_B) \right\} \approx \frac{1}{\beta} \ln \left( e^{\beta \epsilon_F} \right)
\]

\[
\approx \epsilon_F
\]
And so,

\[
M = \frac{\rho_0}{2A\beta} \ln\left( \frac{1 + e^{\beta(\epsilon_B + \mu)}}{1 + e^{\beta(-\epsilon_B + \mu)}} \right) 
\approx \frac{\rho_0}{2A\beta} \ln\left( \frac{1 + e^{\beta(\epsilon_B + \epsilon_F)}}{1 + e^{\beta(-\epsilon_B + \epsilon_F)}} \right) 
\approx \frac{\mu_B \rho_0}{2A\beta} \ln(e^{2\beta\epsilon_B}) 
= \frac{\mu_B \rho_0}{A} \epsilon_B
\]

Meaning,

\[
M \approx \frac{\mu_B^2 \rho_0}{A} B
\]

And so,

\[
\chi_p = \frac{\mu_B^2 \rho_0}{A}
\]