University of Strasbourg

Exam — Session 1

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1 The Blume–Capel model [14 points]

The Blume–Capel model describes a magnetic material with some nonmagnetic vacancies. Let us consider a lattice [we denote by N ($\gg 1$) the number of lattice sites and by z the number of nearest neighbors] of spins S_i that can take the values -1, 0 and $+1$. A spin 0 corresponds to a vacancy (nonmagnetic impurity or empty site) and spins $+1$ or -1 correspond to the two different orientations of the magnetic species. We assume that the Hamiltonian of the system in presence of an homogeneous magnetic field h is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j + \Delta \sum_{i=1}^{N} S_i^2 - h \sum_{i=1}^{N} S_i \tag{1.1}$$

where $J > 0$ is the exchange interaction and where Δ is a constant that can be either negative or positive. In the Hamiltonian above, $\langle i, j \rangle$ denotes a summation over nearest neighbors.

1.1 General discussion

(a) Justify that $-\Delta$ is the energy of creation of a vacancy. In which case ($\Delta > 0$ or $\Delta < 0$) is it favorable to create a vacancy ?

It is favorable to create a vacancy when there is a diminution of energy when one is created, meaning that $+\Delta$ is negative.

(b) At $T = 0$ and $h = 0$, calculate the energy of the system in the three different states $\langle S_i \rangle = +1$, $\langle S_i \rangle = -1$, and $\langle S_i \rangle = 0$. Which state is selected at $T = 0$?

$$\begin{aligned}\langle \mathcal{H} \rangle &= -\left\langle J \sum_{\langle i,j \rangle} S_i S_j \right\rangle + \left\langle \Delta \sum_{i=1}^{N} S_i^2 \right\rangle \cancel{- h \sum_{i=1}^{N} S_i} \\ &= -J \sum_{\langle i,j \rangle} \langle S_i S_j \rangle + \Delta \sum_{i=1}^{N} \langle S_i^2 \rangle\end{aligned}$$

$$C_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

In a mean-field-approximation we neglect correlations, meaning,

$$C_{ij} = 0 \Longrightarrow \langle S_i S_j \rangle = \langle S_i \rangle \langle S_j \rangle$$

$$\langle \mathcal{H} \rangle = -J \sum_{\langle i,j \rangle} \langle S_i \rangle \langle S_j \rangle + \Delta \sum_{i=1}^{N} \langle S_i \rangle \langle S_i \rangle$$

$$\langle \mathcal{H} \rangle_{+1} = -J \sum_{\langle i,j \rangle} 1 + \Delta \sum_{i=1}^{N} 1 = -\frac{1}{2} JNz + \Delta N = N \left(\Delta - \frac{Jz}{2} \right)$$

$$\langle \mathcal{H} \rangle_{-1} = -J \sum_{\langle i,j \rangle} 1 + \Delta \sum_{i=1}^{N} 1 = -\frac{1}{2} JNz + \Delta N = N \left(\Delta - \frac{Jz}{2} \right)$$

$$\langle \mathcal{H} \rangle_{0} = -J \sum_{\langle i,j \rangle} 0 + \Delta \sum_{i=1}^{N} 0 = 0$$

The selected state is the one with the lowest energy, meaning that, if $\Delta - \frac{Jz}{2} < 0$ it is the state $\langle S_i \rangle = \pm +1$ that is selected, and if $\Delta - \frac{Jz}{2} > 0$ it is the state $\langle S_i \rangle = 0$ that is selected.

(c) Which limit of Δ corresponds to the usual two-state Ising model ? How would you call the $\Delta = 0$ model ?

For $\Delta = 0$ we recover the usual Ising model.

1.2 Mean-field approximation

We now aim at performing a mean-field approximation (MFA). We write $S_i = m + \delta S_i$, where $m = \langle S_i \rangle$ is the average magnetization.

(a) Define the spin-spin correlation function C_{ij}. What is the value of C_{ij} in the MFA ?

$$\begin{aligned} C_{ij} &= \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \\ &= \langle (m_i + \delta S_i)(m_j + \delta S_j) \rangle - m_i m_j \\ &= \langle \delta S_i \delta S_j \rangle \end{aligned}$$

Within the MFA, we're neglecting the correlation between spin-spin, meaning,

$$C_{ij} = 0$$

(b) Show that within the MFA, it is possible to write the Hamiltonian (1.1) as

$$\mathcal{H} \simeq \frac{1}{2} NzJm^2 - (h + zJm) \sum_{i=1}^{N} S_i + \Delta \sum_{i=1}^{N} S_i^2.$$

$$
\begin{aligned}
\mathcal{H} &= -J\sum_{\langle i,j\rangle} S_iS_j + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= -J\sum_{\langle i,j\rangle}(m_i+\delta S_i)(m_j+\delta S_j) + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= -J\sum_{\langle i,j\rangle}(m_im_j + m_i\delta S_j + m_j\delta S_i + \delta S_i\delta S_j) + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= -J\sum_{\langle i,j\rangle}(m^2 + m(\delta S_j + \delta S_i) + \delta S_i\delta S_j) + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&\approx -J\sum_{\langle i,j\rangle} m^2 - 2J\sum_{\langle i,j\rangle} m\delta S_i + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= -\frac{1}{2}NzJm^2 - 2Jm\sum_{\langle i,j\rangle}(S_i - m) + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= -\frac{1}{2}NzJm^2 - 2Jm\sum_{\langle i,j\rangle} S_i + 2Jm^2\sum_{\langle i,j\rangle}(1) + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= \frac{1}{2}NzJm^2 - 2Jm\sum_{\langle i,j\rangle} S_i + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= \frac{1}{2}NzJm^2 - zJm\sum_{i=1}^{N} S_i + \Delta\sum_{i=1}^{N} S_i^2 - h\sum_{i=1}^{N} S_i \\
&= \frac{1}{2}NzJm^2 - (h+zJm)\sum_{i=1}^{N} S_i + \Delta\sum_{i=1}^{N} S_i^2
\end{aligned}
$$

$h - zJm$ can be seen as an effective field h_{eff}.

(c) Calculate the free energy F within the MFA.

$$
\begin{aligned}
Z &= \sum_{S_1} \cdots \sum_{S_N} e^{-\beta \mathcal{H}} \\
&= \sum_{S_1} \cdots \sum_{S_N} e^{-\beta\left[\frac{1}{2}NzJm^2-(h+zJm)\sum_{i=1}^{N} S_i+\Delta\sum_{i=1}^{N} S_i^2\right]} \\
&= e^{-\frac{1}{2}\beta NzJm^2} \sum_{\{S_k\}} \prod_i e^{\beta(h+zJm)S_i-\beta\Delta S_i^2} \\
&= e^{-\frac{1}{2}\beta NzJm^2} \prod_i \sum_{S_i=+1,0,-1} e^{\beta(h+zJm)S_i-\beta\Delta S_i^2} \\
&= e^{-\frac{1}{2}\beta NzJm^2} \prod_i \left[1+e^{\beta(h+zJm)-\beta\Delta}+e^{-\beta(h+zJm)-\beta\Delta}\right] \\
&= e^{-\frac{1}{2}\beta NzJm^2} \prod_i \left[1+2e^{-\beta\Delta}\cosh(\beta(h+zJm))\right] \\
&= e^{-\frac{1}{2}\beta NzJm^2} \left[1+2e^{-\beta\Delta}\cosh(\beta(h+zJm))\right]^N
\end{aligned}
$$

which makes it possible to find an expression for free energy. Indeed, we have

$$
\begin{aligned}
F &= -k_B T \ln Z \\
&= \frac{NzJm^2}{2} - Nk_BT\ln\left\{1+2e^{-\beta\Delta}\cosh(\beta(h+zJm))\right\}
\end{aligned}
$$

(d) Demonstrate that the average value $m = \langle S_i \rangle$ is given by the expression

$$
m = -\frac{1}{N}\frac{\partial F}{\partial h}
$$

Deduce that, within the MFA, the magnetization obeys the self-consistent equation (SCE)

$$
m = \frac{2\sinh(\beta[h+zJm])}{\exp(\beta\Delta)+2\cosh(\beta[h+zJm])}.
$$

$$
\begin{aligned}
m_i &= \langle S_i \rangle \\
&= \sum_{S_1} \cdots \sum_{S_N} S_i \frac{\Xi_{S_i}}{\Xi} \\
&= \frac{1}{\beta}\frac{\partial \ln \Xi}{\partial h_i} \\
&= -\frac{\partial F}{\partial h_i}
\end{aligned}
$$

Since we take the mean value of m_i's,

$$m = \frac{\sum_i m_i}{N} = -\frac{1}{N}\frac{\partial F}{\partial h}$$

$$\begin{aligned}
m &= -\frac{1}{N}\frac{\partial F}{\partial h} \\
&= -\frac{1}{N}\frac{\partial}{\partial h}\left\{\frac{NzJm^2}{2} - Nk_BT\ln\left\{1 + 2e^{-\beta\Delta}\cosh(\beta(h + zJm))\right\}\right\} \\
&= k_BT\frac{\partial}{\partial h}\left\{\ln\left\{1 + 2e^{-\beta\Delta}\cosh(\beta(h + zJm))\right\}\right\} \\
&= k_BT\frac{2e^{-\beta\Delta}\beta\sinh(\beta(h + zJm))}{1 + 2e^{-\beta\Delta}\cosh(\beta(h + zJm))} \\
&= \frac{2\sinh(\beta[h + zJm])}{\exp(\beta\Delta) + 2\cosh(\beta[h + zJm])}
\end{aligned}$$

From now on, we consider the case of vanishing magnetic field,$h = 0$.

(e) In the case $\Delta \to -\infty$, discuss the solutions of the SCE.

For $\Delta \to -\infty$,

$$m = \tanh[\beta zJm]$$

We can define a critical temperature,

$$k_BT_c = zJ$$

If $T > T_c$, there is only one solution : $m = 0$, if $T < T_c$, there are three solutions : $m = 0$, $m = \pm m(T)$. Ultimately, if $T = T_c$, there is only one solution : $m = 0$.

(f) In the general case, show that $m = 0$ is a solution of the SCE.

$$m = \frac{2\sinh(\beta[zJm])}{\exp(\beta\Delta) + 2\cosh(\beta[zJm])}$$

We know that $\sinh(0) = 0$ and $\cosh(0) = 1$, so, $m = 0$ is always a solution because :

$$0 = \frac{0}{e^{\beta\Delta} + 2}$$

Hence, it is always a solution.

(g) We now aim at discussing graphically the solutions of the SCE. We define $t = k_BT/zJ$ and $\delta = \Delta/zJ$.

(i) Express the SCE in term of the function

$$f(m) = \frac{2\sinh(m/t)}{\exp(\delta/t) + 2\cosh(m/t)}.$$

$$\begin{aligned} m &= \frac{2\sinh(\beta[zJm])}{\exp(\beta\Delta) + 2\cosh(\beta[zJm])} \\ &= \frac{2\sinh(zJm/k_BT)}{\exp(\Delta/k_BT) + 2\cosh(zJm/k_BT)} \\ &= \frac{2\sinh(m/t)}{\exp(\delta/t) + 2\cosh(m/t)} \qquad \frac{\delta}{t} = \frac{\Delta}{k_BT} \end{aligned}$$

(ii) What is the value of $f(0)$?

$$f(0) = 0$$

(iii) What are the limits of $f(m)$ when $m \to \pm\infty$?

$$\begin{aligned} \lim_{m\to-\infty} f(m) &= \lim_{m\to-\infty} \frac{\cancel{e^{m/t}} - e^{-m/t}}{e^{\delta/t} + \cancel{e^{m/t}} + e^{-m/t}} \\ &= \lim_{m\to-\infty} \frac{-e^{-m/t-\delta/t}}{1 + e^{-m/t-\delta/t}} \\ &\approx -1 \end{aligned}$$

$$\begin{aligned} \lim_{m\to+\infty} f(m) &= \lim_{m\to+\infty} \frac{e^{m/t} \cancel{- e^{-m/t}}}{e^{\delta/t} + e^{m/t} \cancel{+ e^{-m/t}}} \\ &= \lim_{m\to+\infty} \frac{e^{m/t-\delta/t}}{1 + e^{m/t-\delta/t}} \\ &\approx 1 \end{aligned}$$

(iv) Calculate

$$\left.\frac{\mathrm{d}f}{\mathrm{d}m}\right|_{m=0}$$

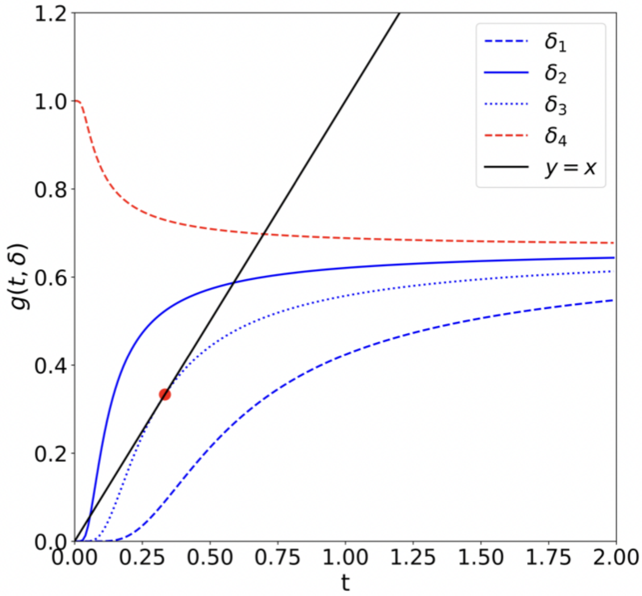


Figure (1) – Colored lines : Plot of $g(t, \delta) = 2/[2 + \exp(\delta/t)]$ as a function of t for different values δ_i of δ. Black solid line : t.

and discuss graphically the number of solutions to the SCE. Show that there is a critical reduced temperature t_c defined by the equation

$$t_c = \frac{2}{2 + \exp(\delta/t_c)}.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}m} f(m) &= \frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{2\sinh(m/t)}{\exp(\delta/t) + 2\cosh(m/t)}\right)\Bigg|_{m=0} \\
&= \frac{\frac{2}{t}\cosh(m/t)(\exp(\delta/t) + 2\cosh(m/t)) - 2\sinh(m/t)\left(\frac{2}{t}\sinh(m/t)\right)}{(\exp(\delta/t) + 2\cosh(m/t))^2}\Bigg|_{m=0} \\
&= \frac{\frac{2}{t}\cancel{(\exp(\delta/t) + 2)}}{(\exp(\delta/t) + 2)^{\cancel{2}}} \\
&= \frac{2/t}{\exp(\delta/t) + 2}
\end{aligned}$$

But the left-side is also equal to 1, so

$$1 = \frac{2/t}{\exp(\delta/t) + 2} \iff t = \frac{2}{2 + \exp(\delta/t)}$$

(v) In figure 1 (colored lines) is plotted the function $g(t, \delta) = 2/[2+\exp(\delta/t)]$ as a function of t for different values δ_i of δ. Which δ_i's are positive and which of them are negative ? Sort by ascending order the δ_i's.

$$g(t,\delta) = \frac{2}{2+\exp(\delta/t)}$$

if δ is positive, when $t \to 0$, the exponential goes to ∞ meaning that $g(t,\delta) = 0$. If δ is negative, when $t \to 0$, the exponential goes to 0 meaning that $g(t,\delta) = 2/2 = 1$.

So looking at the curves, the red one (δ_4) is for a negative δ and the three others are positive.

(vi) Plot the curve $g(t,\delta)$ for the value of δ corresponding to the Ising model and give the corresponding t_c.

For the Ising model,

$$\delta = \frac{\Delta}{zJ} = 0$$

So,

$$g(t,\delta) = 1$$

Meaning $t_c = 1$.

(vii) Using your previous discussion and question 1.1(b), sketch the general behavior of t_c as a function of δ.

2 Pauli paramagnetism of a two-dimensional electron gas [6 points]

In this exercice, we wish to understand one of the magnetic properties of a noninteracting electron gas : the Pauli paramagnetism which is due to the alignment of the electronic magnetic moments with the applied magnetic field. The one-electron Hamiltonian describing this phenomenon is given by

$$H = \frac{\boldsymbol{p}^2}{2m} - \mu - zB. \tag{2.1}$$

Here, $\boldsymbol{p}$ is the electron momentum, m its mass, $\mu_z = qS_z/m$ its magnetic moment, which is related to its spin S_z through the gyromagnetic factor $\gamma = q/m$, where $q = -e$ (with $e = 1.6 \times 10^{-19}$ C the elementary charge). In what follows, we consider a homogeneous magnetic field B parallel to the z axis, and we assume that electrons are confined to a two-dimensional rectangular surface with area $\mathcal{A} = L_x L_y$, where L_x and L_y are the lateral dimensions of the electron gas in the x and y directions, respectively.

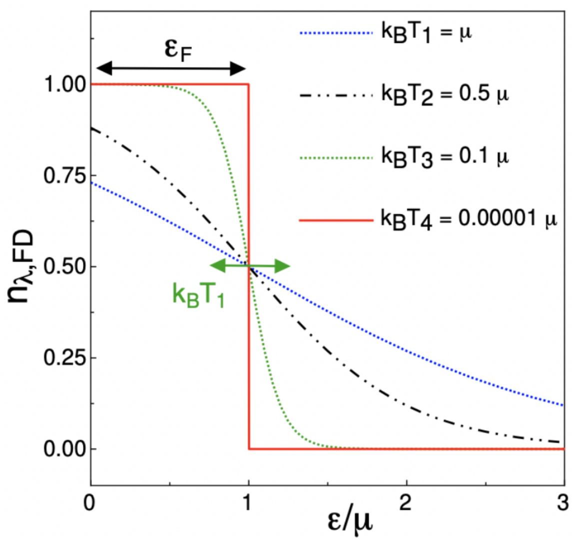
We recall that electrons are spin 1/2 particles, so that they obey the Fermi–Dirac statistics. The average occupancy of an energy state ϵ is then given by

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \tag{2.2}$$

where $\beta = 1/k_B T$, with T the temperature of the gas, and where $\mu = \mu(T)$ is the chemical potential.

2.1 Warm up

(a) Plot the Fermi–Dirac distribution (2.2) for both $T = 0$ and $T \neq 0$.



(b) How is defined the Fermi energy ϵ_F in terms of μ?

$$\epsilon_F = \mu(T = 0)$$

(c) In absence of magnetic field, the Hamiltonian (2.1) reduces to

$$H = \frac{\boldsymbol{p}^2}{2m} \tag{2.3}$$

Using periodic boundary conditions, one can easily show that the spectrum corresponding to the Hamiltonian (2.3) is given by $\epsilon_{\boldsymbol{k}} = \hbar^2|\boldsymbol{k}|^2/2m$, where the wavevector $\boldsymbol{k} = (k_x, k_y)$ is quantized according to $k_x = 2\pi n_x/L_x$ and $k_y = 2\pi n_y/L_y$, with n_x and n_y integer numbers. Show that the corresponding density of states (including the spin degeneracy) is energy independent and is given in the thermodynamic limit by

$$\rho_0 = \frac{m\mathcal{A}}{\pi\hbar^2}. \tag{2.4}$$

$$\psi(x, y) = \psi_x(x) \times \psi_y(y)$$

And so in Schödinger's equation,

$$-\frac{\hbar^2}{2m}\left(\psi_x''(x)\psi_y(y) + \psi_x(x)\psi_y''(y)\right) =$$

So,

$$\frac{\psi_x''(x)}{\psi_x(x)} + \frac{\psi_y''(y)}{\psi_y(y)} = -\frac{2mE}{\hbar^2} = k_x^2 + k_y^2$$

And so we have the equation,

$$\psi_i''(x_i) + k_i^2\psi_i(x_i) = 0$$

$$\psi_i(x_i) = \text{Cst} \times e^{ik_ix}$$

Periodic boundary conditions,

$$\psi_i(0) = \psi_i(L_i) \Longrightarrow 1 = e^{ik_iL_i}$$

And so,

$$k_iL_i = 2\pi n_i \Longrightarrow k_i = \frac{2\pi}{L_i}n_i, \qquad n_i \in \mathbb{Z}$$

$$\Omega(0 \to E) = \frac{2\pi|k|^2}{\frac{(2\pi)^2}{L_x L_y}} = \frac{2}{4\pi} L_x L_y \frac{2mE}{\hbar^2} = \frac{\mathcal{A}mE}{\pi\hbar^2}$$

The factor 2 in here is for the spin degeneracy. And so,

$$\rho(E) = \frac{\mathrm{d}\Omega(0 \to E)}{\mathrm{d}E} = \frac{\mathcal{A}m}{\pi\hbar^2}$$

(d) Still in absence of a magnetic field, show that $\epsilon_F = N/\rho_0$, where N is the total number of electrons in the two-dimensional gas.

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} \qquad k_F = \sqrt{\frac{2m\epsilon_F}{\hbar^2}}$$

$$\begin{aligned} N &= \frac{k_F^2}{\frac{(2\pi)^2}{L_x L_y}} \\ &= \frac{\epsilon_F m \mathcal{A}}{\pi\hbar^2} \\ &= \epsilon_F \rho_0 \end{aligned}$$

And so,

$$\epsilon_F = \frac{N}{\rho_0}$$

2.2 Pauli paramagnetism

The energy spectrum corresponding to the Hamiltonian (2.1) is spin dependent, and given by

$$\epsilon_{\boldsymbol{k}}^{\pm} = \frac{\hbar^2|\boldsymbol{k}|^2}{2m_*} \mp \epsilon_B,$$

where + (−) corresponds to a spin up (down) electron. Here, $\epsilon_B = \mu_B B$, with $\mu_B = \hbar q/2m$ the Bohr magneton.

(a) Show that the density of states of the two spin species is energy dependent and given by

$$\rho_{\pm}(\epsilon) = \frac{1}{2}\rho_0 \theta(\epsilon \pm \epsilon_B),$$

where $\theta(x)$ is the Heaviside step function.

At $T = 0$,

$$\epsilon^+ = \epsilon_k - \epsilon_B = \frac{\hbar^2 k^2}{2m} - \epsilon_B$$

$$\rho_+(\epsilon)\mathrm{d}\epsilon = \frac{1}{2}\frac{\mathcal{A}m}{\pi\hbar^2}(\epsilon + \epsilon_B)\mathrm{d}\epsilon$$

And we know that $\rho_+(\epsilon)\mathrm{d}\epsilon > 0$, implying that $\epsilon + \epsilon_B > 0$. But, if $\epsilon + \epsilon_B < 0$, this does mean that $\hbar^2 k^2/2m < 0$, which is impossible, meaning there is not states like that and so we get,

$$\begin{aligned}\rho_+(\epsilon) &= \frac{1}{2}\frac{\mathcal{A}m}{\pi\hbar^2}\theta(\epsilon + \epsilon_B) \\ &= \frac{1}{2}\rho_0\theta(\epsilon + \epsilon_B)\end{aligned}$$

And same for ρ_- so we get,

$$\rho_\pm(\epsilon) = \frac{1}{2}\rho_0\theta(\epsilon \pm \epsilon_B)$$

where the $1/2$ factor is here because of the lift of the degeneracy from the without field case.

(b) Let us first assume that both the temperature T and the chemical potential μ are fixed. Show that the average number of spin up and spin down electrons, denoted by $N_\pm$, is given by

$$N_\pm = \frac{\rho_0}{2\beta}\ln\left(1 + e^{\beta[\pm\epsilon_B + \mu]}\right).$$

$$\begin{aligned}N_\pm &= \int \mathrm{d}\epsilon\rho_\pm(\epsilon)f(\epsilon) \\ &= \int \mathrm{d}\epsilon\frac{1}{2}\rho_0\theta(\epsilon \pm \epsilon_B)\frac{1}{e^{\beta(\epsilon-\mu)} + 1}\end{aligned}$$

$$\begin{aligned}N_+ &= \frac{1}{2}\rho_0\int_{-\epsilon_B}^{+\infty}\mathrm{d}\epsilon\frac{1}{e^{\beta(\epsilon-\mu)} + 1} \\ &= \frac{1}{2}\rho_0\int_{-\epsilon_B}^{+\infty}\mathrm{d}\epsilon\frac{e^{-\beta(\epsilon-\mu)}}{e^{-\beta(\epsilon-\mu)} + 1} \\ &= -\frac{1}{\beta 2}\rho_0\int_{-\epsilon_B}^{+\infty}\mathrm{d}\epsilon\frac{-\beta e^{-\beta(\epsilon-\mu)}}{e^{-\beta(\epsilon-\mu)} + 1} \\ &= -\frac{\rho_0}{2\beta}\left[\ln\left(1 + e^{-\beta(\epsilon-\mu)}\right)\right]_{-\epsilon_B}^{+\infty} \\ &= \frac{\rho_0}{2\beta}\ln\left(1 + e^{\beta(\epsilon+\mu)}\right)\end{aligned}$$

$$N_- = \int_{-\infty}^{+\infty} \mathrm{d}\epsilon \frac{1}{2}\rho_0 \theta(\epsilon - \epsilon_B) \frac{1}{1+e^{\beta(\epsilon-\mu)}}$$
$$= \frac{\rho_0}{-2\beta} \int_{\epsilon_B}^{+\infty} \frac{-\beta e^{-\beta(\epsilon-\mu)}}{e^{-\beta(\epsilon-\mu)}+1}$$
$$= -\frac{\rho_0}{2\beta} \left[\ln\left(1+e^{-\beta(\epsilon-\mu)}\right)\right]_{\epsilon_B}^{+\infty}$$
$$= \frac{\rho_0}{2\beta} \ln\left(1+e^{-\beta(\epsilon_B-\mu)}\right)$$

In the end,

$$N_\pm = \frac{\rho_0}{2\beta} \ln\left(1+e^{\beta(\pm\epsilon_B+\mu)}\right)$$

(c) Let us now consider that the total number of electrons $N = N_+ + N_-$ is fixed. Deduce from the preceding question a quadratic equation for the fugacity $z = e^{\beta\mu}$. Give the resulting expression of the chemical potential as a function of ϵ_F and ϵ_B. In particular, analyze the low temperature ($\beta\epsilon_F \gg 1$) and low magnetic field ($\beta\epsilon_B \ll 1$) limits.

$$N = N_+ + N_- \qquad\qquad z = e^{\beta\mu}$$

$$N_\pm = \frac{\rho_0}{2\beta} \ln\left(1+ze^{\pm\beta\epsilon_B}\right)$$

$$N = \frac{\rho_0}{2\beta} \ln\left[\left(1+ze^{\beta\epsilon_B}\right)\left(1+ze^{-\beta\epsilon_B}\right)\right]$$
$$= \frac{\rho_0}{2\beta} \ln\left(1+2z\cosh(\beta\epsilon_B)+z^2\right)$$

Meaning,

$$\frac{2\beta N}{\rho_0} = \ln\left(1+2z\cosh(\beta\epsilon_B)+z^2\right)$$

$$z^2 + 2z\cosh(\beta\epsilon_B) + 1 - e^{\frac{2\beta N}{\rho_0}} = 0$$

The solution of this polynomial is

$$z = \frac{-2\cosh(\beta\epsilon_B) + \sqrt{4\cosh^2(\beta\epsilon_B) - 4(1-e^{\frac{2\beta N}{\rho_0}})}}{2} > 0$$

We know that $z = e^{\beta\mu}$ and so,

$$\mu = \frac{1}{\beta}\ln\left\{\frac{-2\cosh(\beta\epsilon_B) + \sqrt{4\cosh^2(\beta\epsilon_B) - 4(1 - e^{\frac{2\beta N}{\rho_0}})}}{2}\right\}$$

$$\lim_{T\to 0,\ B\to 0}\mu = \epsilon_F$$

If $B = 0$, $\epsilon_B = 0$, meaning,

$$\mu_{B=0} = \frac{1}{\beta}\ln\left\{e^{\frac{\beta N}{\rho_0}} - 1\right\}$$

$$\mu_{B=0,\ \beta\to\infty} = \frac{1}{\beta}\ln\left(e^{\frac{\beta N}{\rho_0}}\right) = \frac{N}{\rho_0}$$

And thus we recover

$$\epsilon_F = \frac{N}{\rho_0}$$

(d) The magnetization of the electron gas is given by $M = \mu_B(N_+ - N_-)/\mathcal{A}$, and the corresponding susceptibility is defined as

$$\chi_P = \lim_{B\to 0}\frac{\partial M}{\partial B}.$$

Calculate the Pauli susceptibility χ_P as a function of ρ_0, $\mathcal{A}$, and μ_B in the degenerate limit $\beta\epsilon_F \gg 1$.

$$N_+ = \frac{\rho_0}{2\beta}\ln\big(1 + e^{\beta(\epsilon_B+\mu)}\big) \qquad N_- = \frac{\rho_0}{2\beta}\ln\big(1 + e^{\beta(-\epsilon_B+\mu)}\big)$$

$$N_+ - N_- = \frac{\rho_0}{2\beta}\ln\left(\frac{1 + e^{\beta(\epsilon_B+\mu)}}{1 + e^{\beta(-\epsilon_B+\mu)}}\right)$$

But we are at $\beta\epsilon_F \gg 1$,

$$\mu = \frac{1}{\beta}\ln\left\{\sqrt{\cosh^2(\beta\epsilon_B) + e^{2\beta\epsilon_F} - 1} - \cosh(\beta\epsilon_B)\right\} \qquad \approx \frac{1}{\beta}\ln\big(e^{\beta\epsilon_F}\big)$$

$$\approx \epsilon_F$$

And so,

$$\begin{aligned} M &= \frac{\rho_0}{2\mathcal{A}\beta} \ln\left(\frac{1 + e^{\beta(\epsilon_B + \mu)}}{1 + e^{\beta(-\epsilon_B + \mu)}} \right) \\ &\approx \frac{\rho_0}{2\mathcal{A}\beta} \ln\left(\frac{1 + e^{\beta(\epsilon_B + \epsilon_F)}}{1 + e^{\beta(-\epsilon_B + \epsilon_F)}} \right) \\ &\approx \frac{\mu_B \rho_0}{2\mathcal{A}\beta} \ln\left(e^{2\beta\epsilon_B} \right) \\ &= \frac{\mu_B \rho_0}{\mathcal{A}} \epsilon_B \end{aligned}$$

Meaning,

$$M \approx \frac{\mu_B^2 \rho_0}{\mathcal{A}} B$$

And so,

$$\chi_p = \frac{\mu_B^2 \rho_0}{\mathcal{A}}$$