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Problem Set 2
Quantum statistics

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Reminder

1 Two-dimensional electron gas

A confined electron gas can form at the interface between two doped semiconductors (e.g., GaAs/AlGaAs). The confinement is such that one can consider that the gas is strictly two-dimensional. Electron-electron interactions will be neglected in the following and we will adopt the effective mass approximation. We call n the electronic density of the gas and $A = L_x L_y$ its surface (which we assume to be very large as compared to all the other length scales of the problem). Here, L_x and L_y are the lateral dimensions of the gas in the x and y directions, respectively. We recall that the electrons are spin-1/2 fermions, and thus obey the Fermi–Dirac statistics. The average occupancy of an energy state ϵ is then given by the Fermi–Dirac distribution function

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}, \tag{1}$$

where $\beta = 1/k_B T$, with T the temprature of the gas, and where $\mu = \mu(T)$ is the chemical potential.

(a) Plot the Fermi–Dirac distribution (1). In particular, analyze the $T = 0$ case.

The Fermi-Dirac distribution is a descending function, in this case,

$$f(+\infty) = 0$$

At $T = 0$ there is two options :

$$\epsilon - \mu > 0 \longrightarrow f(\epsilon) = 0$$
$$\epsilon - \mu < 0 \longrightarrow f(\epsilon) = 1$$

(b) Using periodic boundary conditions (why can you do so ?), solve Schrödinger's equation and show that the electronic dispersion is given by

$$\epsilon_{\boldsymbol{k}} = \frac{\hbar^2 |\boldsymbol{k}|^2}{2m},$$

where the wavevector $\boldsymbol{k} = (k_x, k_y)$ is quantized according to $k_x = 2\pi n_x / L_x$ and $k_y = 2\pi n_y / L_y$, with n_x and n_y integer numbers.

Time-dependent Schrödinger's equation :

$$-\frac{\hbar^2}{2m}\frac{\partial \psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\Delta\psi$$

Time-independent Schrödinger's equation :

$$\hat{H}\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=E\psi$$

We search ψ as the expression

$$\psi(x,y)=Ae^{ik_x x}e^{ik_y y}$$

with the boundary conditions

$$\frac{\hbar^2}{2m}\left(k_x^2+k_y^2\right)=\epsilon_{\boldsymbol{k}}\Longrightarrow\epsilon_{\boldsymbol{k}}=\frac{\hbar^2|\boldsymbol{k}|^2}{2m}$$

> We have to check the conditions, **Appendix 2 from the Diu** : it has to deal with the boundaries effects : they are negligeable. We do use the periodic limit conditions. Equivalent to hard-wall boundaries if the system is macroscopic. The boundaries effects do play a negligeable role statisticaly. The Periodic-Limit-Condition (PLC) :
>
> $$\psi(x+L,y)=\psi(x,y)\Longrightarrow k_x=\frac{2\pi}{L}n_x$$
>
> with $n_x\in\mathbb{Z}$.

(c) Show that the electronic density of states $\rho(\epsilon)$ is energy-independent and is given by $\rho(\epsilon)=1/\Delta$, where $\Delta=\pi\hbar^2/m\mathcal{A}$.

We know that the density of states is expressed as

$$\rho(\epsilon)\mathrm{d}\epsilon=\sum_\lambda\delta(\epsilon-\epsilon_\lambda)$$

> **Remark** : to convice yourself of that expression, one could write the average of an observable,
>
> $$\langle A\rangle=\sum_\lambda A_\lambda f_\lambda=\int_0^{+\infty}\mathrm{d}\epsilon\sum_\lambda\delta(\epsilon-\epsilon-\lambda)A(\epsilon)f(\epsilon)=\int_0^{+\infty}\mathrm{d}\epsilon\rho(\epsilon)A(\epsilon)f(\epsilon)$$
>
> and see the expression of the density of state appear.

$$\begin{aligned}
\rho(\epsilon) &= \sum_{\boldsymbol{k}} \delta(\epsilon - \epsilon_{\boldsymbol{k}}) \times 2 && \text{factor 2 is for the degeneracy} \\
&\approx \int \mathrm{d}^2\boldsymbol{k} \frac{1}{\frac{2\pi}{L_x}\frac{2\pi}{L_y}} \delta\left(\epsilon - \frac{\hbar^2 k^2}{2m}\right) && \text{because energy gap small compared to 1} \\
&= \frac{\mathcal{A}}{2\pi^2} \int_0^{+\infty} \mathrm{d}k k \int_0^{2\pi} \mathrm{d}\theta \delta\left(\epsilon - \frac{\hbar^2 k^2}{2m}\right) && \text{change of variable} \\
&= \frac{\mathcal{A}}{\pi} \frac{m}{\hbar^2} \int_0^{+\infty} \mathrm{d}E \delta(\epsilon - E) \\
&= \frac{\mathcal{A}}{\pi} \frac{m}{\hbar^2}
\end{aligned}$$

Meaning,

$$\rho(\epsilon) = \frac{1}{\Delta} \qquad \Delta = \frac{\pi \hbar^2}{m\mathcal{A}}$$

Where Δ can be seen as the mean energy spacing.

(d) Give an expression for the average number N of electrons in the gas. Deduce from the previous result that the chemical potential reads

$$\mu(T) = k_B T \ln\left(e^{T_F/T} - 1\right),$$

where T_F is the Fermi temperature, defined through the Fermi energy as $E_F = k_B T_F$. What is the definition of the Fermi energy ? Give an expression of E_F as a function of N and Δ. Interprete this result. Plot μ as a function of T.

$$\begin{aligned}
\langle N \rangle &= \sum_\lambda f_\lambda \\
&= \int_0^{+\infty} \mathrm{d}\epsilon \sum_\lambda \delta(\epsilon - \epsilon_\lambda) f(\epsilon) \\
&= \int_0^{+\infty} \mathrm{d}\epsilon \rho(\epsilon) f(\epsilon) \\
&\approx \frac{1}{\Delta} \int_0^{+\infty} \frac{1}{e^{\beta(\epsilon-\mu)} + 1} && X = \beta(\epsilon - \mu) \\
&= \frac{1}{\beta\Delta} \int_{-\beta\mu}^{+\infty} \mathrm{d}X \frac{1}{e^X + 1} \times \frac{e^{-X}}{e^{-X}} \\
&= -\frac{1}{\beta\Delta} \left[\ln\left(1 + e^{-X}\right)\right]_{-\beta\mu}^{+\infty} \\
&= \frac{1}{\beta\Delta} \ln\left(1 + e^{\beta\mu}\right)
\end{aligned}$$

So we can invert that equality,

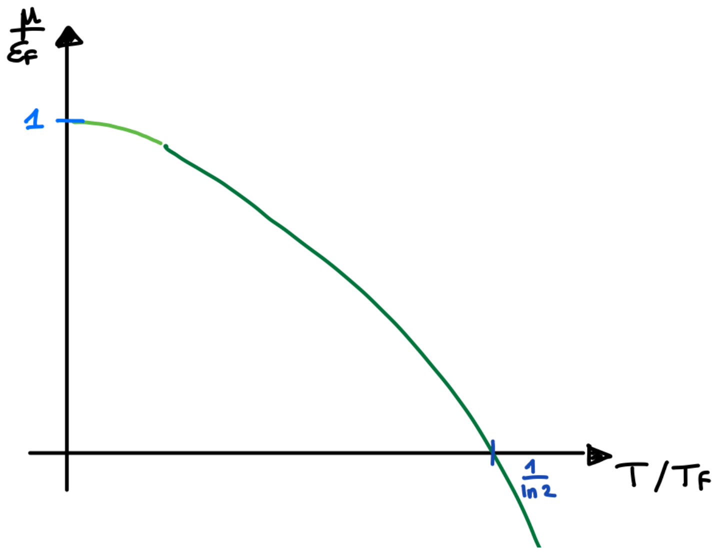
$$\mu = k_B T \ln\bigl(e^{N\Delta/k_B T} - 1\bigr) k_B T$$

- $T \ll T_F$:

$$\frac{\mu}{\epsilon_F} = \frac{T}{T_F}\ln\bigl(e^{T_F/T} - 1\bigr) = \frac{T}{T_F}\ln\bigl(e^{T_F/T}\left[1 - e^{-T_F/T}\right]\bigr) \approx 1 - \frac{T}{T_F}\left(e^{-T_F/T}\right)$$

- $T \gg T_F$:

$$\frac{\mu}{\epsilon_F} \approx \frac{T}{T_F}\ln\left(\frac{T_F}{T}\right)$$



For most metal their fermi temperature T_F is huge compared to T, hence they are on the light green part of the plot.

(e) Give a formal expression of the average energy E of the system in terms of an integral over ϵ, that we will not explicitly calculate. Show that the grand-canonical potential reads

$$\Omega = -\frac{k_B T}{\Delta}\int_0^\infty \mathrm{d}\epsilon \ln\bigl(1 + e^{-\beta(\epsilon-\mu)}\bigr)$$

Deduce from the previous two results that $\Omega = -E$.

$$\begin{aligned} E &= \sum_\lambda \epsilon_\lambda f_\lambda \\ &= \int_0^{+\infty} \mathrm{d}\epsilon \rho(\epsilon) f(\epsilon)\epsilon \\ &= \frac{1}{\Delta}\int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon}{e^{\beta(\epsilon-\mu)} + 1} \end{aligned}$$

Let's leave it like that for now.

$$\Omega = -k_B T \ln \Xi$$

Where Ξ is the grand canonical partition function

$$\begin{aligned}
\Xi &= \sum_{(l)} e^{-\beta(\epsilon_l - \mu N_l)} \\
&= \sum_{n_{\lambda_1}=0}^{1} \sum_{n_{\lambda_2}=0}^{1} \cdots e^{-\beta(\sum_{\lambda_1,\lambda_2\cdots} \epsilon_\lambda n_\lambda - \mu \sum_{\lambda_1,\lambda_2\cdots} n_\lambda)} \\
&= \sum_{n_{\lambda_1}=0}^{1} \sum_{n_{\lambda_2}=0}^{1} \cdots \prod_{\lambda_1,\lambda_2\cdots} e^{-\beta(\epsilon_\lambda - \mu) n_\lambda} \\
&= \prod_{\lambda} \sum_{n_\lambda=0}^{1} e^{-\beta(\epsilon_\lambda - \mu) n_\lambda}
\end{aligned}$$

Meaning,

$$\Xi = \prod_{\lambda} \left(1 + e^{-\beta(\epsilon_\lambda - \mu)}\right)$$

Meaning,

$$\begin{aligned}
\Omega &= -k_B T \sum_{\lambda} \ln\left(1 + e^{-\beta(\epsilon_\lambda - \mu)}\right) \\
&= -\frac{k_B T}{\Delta} \int_0^{+\infty} \mathrm{d}\epsilon \cdot 1 \cdot \ln\left(1 + e^{-\beta(\epsilon_\lambda - \mu)}\right) && \text{integration by parts} \\
&= \cancel{\left[-\frac{k_B T}{\Delta} \epsilon \ln\left(1 + e^{-\beta(\epsilon_\lambda - \mu)}\right)\right]_0^{+\infty}} + \frac{k_B T}{\Delta} \int_0^{+\infty} \mathrm{d}\epsilon \epsilon (-\beta) \frac{e^{-\beta(\epsilon - \mu)}}{1 + e^{-\beta(\epsilon - \mu)}} \\
&= -E
\end{aligned}$$

So,

$$\Omega = -E$$

(f) Show that the two-dimensional pressure P of the gas is related to the average energy via the expression $P = E/A$.

$$P = -\frac{\partial \Omega}{\partial \mathcal{A}}$$

From thermodynamics second principle,

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}\mathcal{A} + \mu\mathrm{d}N$$

$$\Omega = F - \mu N = E - TS - \mu N$$

$$\mathrm{d}\Omega = -p\mathrm{d}\mathcal{A} - S\mathrm{d}T - N\mathrm{d}\mu$$

And using previous results,

$$P = +\frac{\partial E}{\partial \mathcal{A}} = \frac{E}{\mathcal{A}}$$

(g) Using your answers to questions (e) and (f) above, derive the equation of state at $T = 0$.

$$P = \frac{1}{\Delta\mathcal{A}}\int_0^{\epsilon_F} \mathrm{d}\epsilon\epsilon = \frac{\epsilon_F^2}{2\Delta\mathcal{A}}$$

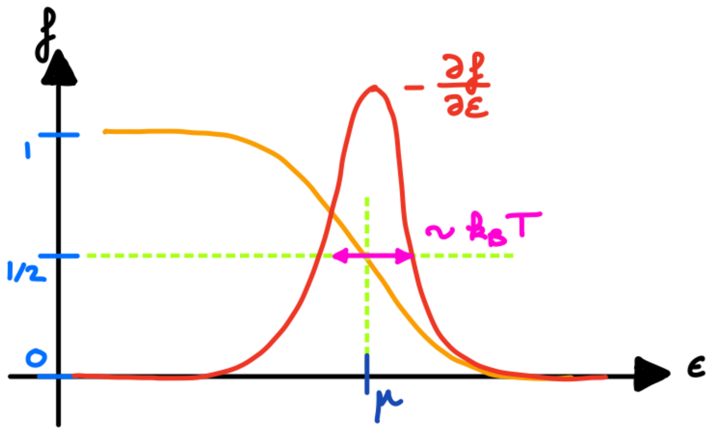
$$P = \frac{N^2\Delta^2}{2\Delta\mathcal{A}} = \frac{N^2\pi\hbar^2}{m\mathcal{A}2\mathcal{A}} = \frac{\pi\hbar^2}{2m}\left(\frac{N}{\mathcal{A}}\right)^2$$

The pressure is not zero at temperature $T = 0$ because the fermions have a repulsive quantum intereactifon due to the fermi exclusion principle.

(h) At low temperature ($T \ll T_F$), expand the average energy to second order in T/T_F so as to obtain the equation of state. Notice that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \frac{x^2 e^x}{(e^x+1)^2} = \frac{\pi^2}{3}.$$

$$E = \int_0^{+\infty} \mathrm{d}\epsilon\rho(\epsilon)\epsilon f(\epsilon) = \frac{1}{\Delta}\int_0^{+\infty} \mathrm{d}\epsilon\epsilon f(\epsilon)$$



$$E = \cancel{\left[\frac{1}{\Delta}\frac{\epsilon^2}{2}f(\epsilon)\right]_0^{+\infty}} + \frac{1}{\Delta}\int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon^2}{2}\left(-\frac{\partial f}{\partial \epsilon}\right)$$

Performing a taylor expansion of ϵ^2 around μ,

$$\epsilon^2 \approx \mu^2 + 2\mu(\epsilon-\mu) + (\epsilon-\mu)^2$$

$$\begin{aligned} E &= \frac{1}{\Delta}\int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon^2}{2}\left(-\frac{\partial f}{\partial \epsilon}\right) \\ &\approx \frac{1}{2\Delta}\int_{-\infty}^{+\infty} \mathrm{d}\epsilon \left[\mu^2 + 2\mu(\epsilon-\mu) + (\epsilon-\mu)^2\right]\left(-\frac{\partial f}{\partial \epsilon}\right) \end{aligned}$$

We change the integral from 0 to $-\infty$ because the function is exponentially close to 0 on the interval $]-\infty, 0]$.

$$\int_{-\infty}^{+\infty} \mathrm{d}\epsilon \left(-\frac{\partial f}{\partial \epsilon}\right) = -f(+\infty) + f(-\infty) = 1$$

$$\int_{-\infty}^{+\infty} \mathrm{d}\epsilon(\epsilon-\mu) \left(-\frac{\partial f}{\partial \epsilon}\right) = 0$$

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \qquad f'(\epsilon) = -\frac{\beta e^{\beta}(\epsilon-\mu)}{[e^{\beta(\epsilon-\mu)}+1]^2}$$

$$\beta\int_{-\infty}^{+\infty} \mathrm{d}\epsilon(\epsilon-\mu)^2 \frac{e^{\beta}(\epsilon-\mu)}{[e^{\beta(\epsilon-\mu)}+1]^2}$$

$x = \beta(\epsilon-\mu)$ and $\mathrm{d}x = \beta \mathrm{d}\epsilon$,

$$\begin{aligned} \beta\int_{-\infty}^{+\infty} \mathrm{d}\epsilon(\epsilon-\mu)^2 \frac{e^{\beta}(\epsilon-\mu)}{[e^{\beta(\epsilon-\mu)}+1]^2} &= \int_{\mathbb{R}} \mathrm{d}x \frac{x^2}{\beta^2}\frac{e^x}{(e^x+1)^2} \\ &= \frac{\pi^2}{3}(k_BT)^2 \end{aligned}$$

Meaning,

$$E \approx \frac{1}{2\Delta}\left[\mu^2 + \frac{\pi^2}{3}(k_BT)^2\right]$$

$$\begin{aligned} \mu(T) &= k_BT\ln\left(e^{T_F/T}-1\right) \\ &= k_BT\ln\left(e^{T_F/T}\left(1-e^{-T_F/T}\right)\right) \\ &= E_F + k_BT\ln\left(1-e^{-T_F/T}\right) \\ &= E_F - k_BTe^{-T_F/T} \\ &\approx E_F \end{aligned}$$

$$E \approx \frac{E_F^2}{2\Delta}\left[1 + \frac{\pi^2}{3}\left(\frac{T}{T_F}\right)^2\right]$$

$$P = \frac{E}{\mathcal{A}} = \frac{\pi\hbar^2 n^2}{2m}\left[1 + \frac{\pi^2}{3}\left(\frac{T}{T_F}\right)^2\right]$$

(i) Calculate the equation of state at high temperature ($T \gg T_F$). Comment your result.

$$\begin{aligned} P = \frac{E}{\mathcal{A}} &= \frac{1}{\Delta\mathcal{A}}\int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon}{e^{\beta(\epsilon-\mu)}+1} \\ &\approx \frac{1}{\Delta\mathcal{A}}\int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon}{e^{\beta\epsilon \frac{T}{T_F}}} \end{aligned}$$

And so we get,

$$P = \frac{Nk_BT}{\mathcal{A}}$$

So we get that for 2D fermions for large temperature, its relation is linear.

(j) *(Optional question)* Calculate now the equation of state for an arbitrary temperature. One gives

$$\int_0^{\infty} \mathrm{d}x \frac{x}{e^x/a+1} = -\mathrm{Li}_2(-a),$$

where a is a constant, and where $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ is the polylogarithm function of order s.

(k) Compare all the results of this problem to the three-dimensional case encountered in the lecture.

2 Bose–Einstein condensation

Let us consider a system of N bosons with mass m and spin s ($s \in \mathbb{N}$) occupying a volume V. In the following, the interactions between the bosons are neglected. Accessible energy levels are denoted by $\epsilon_{\boldsymbol{k}}$, and the ground state energy is set to zero.

(a) What is the average occupancy $n(\epsilon)$ of a state of energy ϵ at temperature T ? Show that the density of states takes the form $d(\epsilon) = KV\sqrt{\epsilon}$, where K is a constant. Give the expression for K. What is the sign of the chemical potential μ ? How is μ determined in the thermodynamic limit ?

- Ω_λ is the grand potential of one state

$$\begin{aligned}\Omega_\lambda &= -k_B T \ln\left(\sum_{n_\lambda=0}^{+\infty} e^{-\beta(\epsilon_\lambda-\mu)n_\lambda}\right)\\ &= -k_B T \ln\left(\frac{1}{1-e^{-\beta(\epsilon_\lambda-\mu)}}\right)\\ &= -k_B T \ln\left(1-e^{-\beta(\epsilon_\lambda-\mu)}\right)\end{aligned}$$

We also know that

$$N_\lambda = -\frac{\partial \Omega_\lambda}{\partial \mu}$$

So,

$$n(\epsilon) = -k_B T \frac{\beta e^{-\beta(\epsilon_\lambda-\mu)}}{1-e^{-\beta(\epsilon_\lambda-\mu)}} = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

Which is the average occupation $n(\epsilon)$ of a state of energy ϵ.

- Density of states

$$s\epsilon_k = \frac{\hbar^2 k^2}{2m} \qquad\qquad \boldsymbol{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

with $n_\alpha \in \mathbb{Z}$, $\alpha \in \{x, y, z\}$. This is valid for one particule.

$$\begin{aligned}\psi(k_0) = \text{number of accessible states} &= g_s \frac{4}{3}\pi |k_0|^2 \frac{1}{\left(\frac{2\pi}{L}\right)^3}\\ &= g_s \frac{V k_0^3}{6\pi^2}\\ &= \frac{g_s V}{6\pi^2}\left(\frac{2m\epsilon}{\hbar^2}\right)^{3/2}\end{aligned}$$

For bosonic interaction, the particule labeled 1 on the state 1 and the particule labeled 2 on the state 2 is equivalent to the particule labeled 1 on the state 2 and the particule labeled 2 on the state 1.

$$\begin{aligned}\mathrm{d}(\epsilon) &= \frac{3}{2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2s+1}{6\pi^2}V\sqrt{\epsilon}\\ &= KV\sqrt{\epsilon}\end{aligned}$$

So,

$$K = \frac{g_s}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

- Sign of μ

  Remark : $\epsilon_k = \hbar^2\boldsymbol{k}^2/2m$ can be null : the states of energy 0 are accessible, so $n(0)$ must make sense, hence $(e^{-\beta\mu}-1)^{-1}$ must make sense, implying that $e^{-\beta\mu}$ is greater than 1, hence $\mu < 0$.

  A legitimate question arise : In the grand canonical ensemble, it is the experimenter who decides the value. We can say that $\mu > 0$, what does happen ?

$$n(0) \longrightarrow \epsilon_0 = \sum_{n_\lambda=0}^{+\infty} e^{+\beta\mu n_\lambda} = \sum_{n_\lambda\geq 0}\left(e^{\beta\mu}\right)^{n_\lambda} = +\infty$$

  Meaning that we give an increasing statistical weight as n_0 increases.

  There is always a flow of particles that wants to fill the system. We are not at equilibrium, that is to say that a statistical equilibrium is only found if $\mu < 0$. In the canonical ensemble we control N and not μ.

(b) Plot on the same graph $n(\epsilon)$ as a function of ϵ for two different chemical potentials $\mu_1 < \mu_2$ while T is being kept fixed. On another graph, plot $n(\epsilon)$ as a function of ϵ for two different temperatures $T_1 < T_2$ while μ is being kept fixed. Considering that the number of particles is fixed, show that

$$\left(\frac{\partial\mu}{\partial T}\right)_N < 0$$

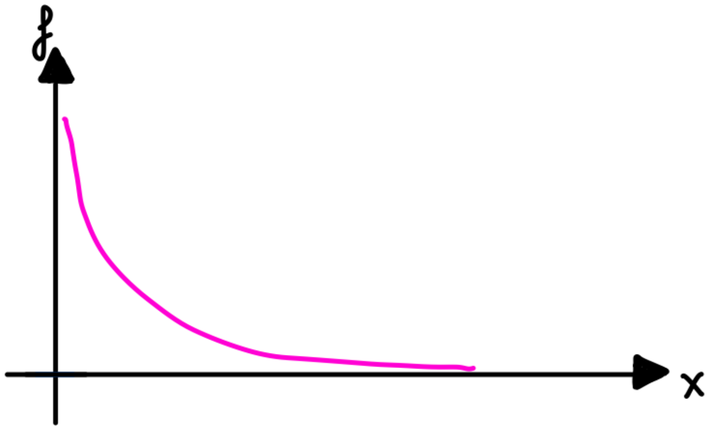
$$f(X) = \frac{1}{e^X - 1}$$

- $X \ll 1$

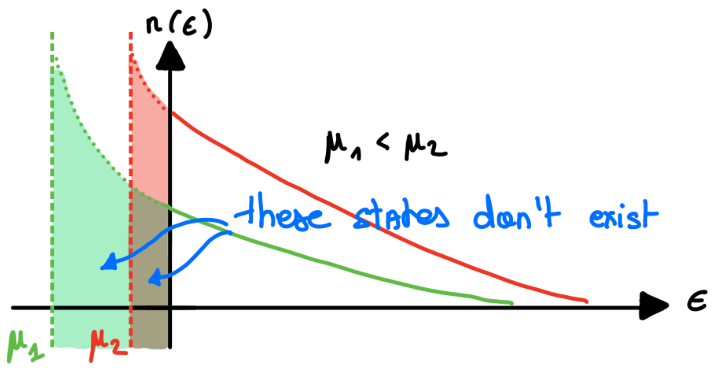
$$f(X) \approx \frac{1}{X}$$

- $X \gg 1$

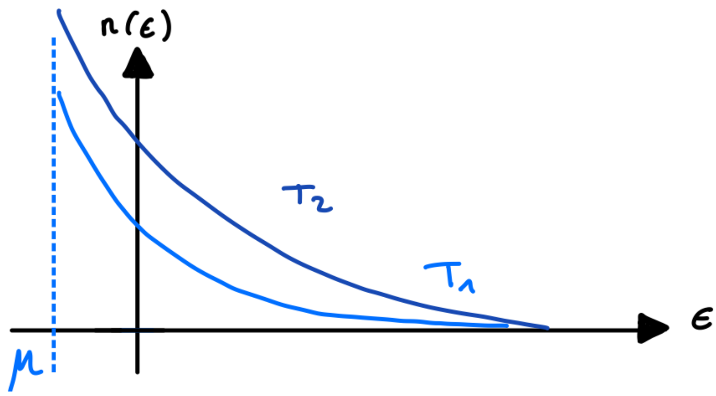
$$f(X) \approx e^{-X}$$



$$\left.\frac{\partial n}{\partial \mu}\right|_T > 0$$



$$\left.\frac{\partial n}{\partial T}\right|_\mu > 0$$



$$N(T,\mu) \longrightarrow \mathrm{d}N = \left.\frac{\partial N}{\partial T}\right|_{\mu} s\mathrm{d}T + \left.\frac{\partial N}{\partial \mu}\right|_{T} \mathrm{d}\mu$$

$$\mathrm{d}N = 0 \longrightarrow \left.\frac{\partial \mu}{\partial T}\right|_{N} = -\frac{\left.\frac{\partial N}{\partial T}\right|_{\mu}}{\left.\frac{\partial N}{\partial \mu}\right|_{T}}$$

But, what is N,

$$N = \int_0^{+\infty} \mathrm{d}\epsilon\, \mathrm{d}(\epsilon)\, n(\epsilon)$$

$$\left.\frac{\partial N}{\partial T}\right|_{\mu} = \int_0^{+\infty} \mathrm{d}\epsilon\, \mathrm{d}(\epsilon) \left.\frac{\partial n}{\partial T}\right|_{\mu} \geq 0$$

> Because $\mathrm{d}(\epsilon) \geq 0$ and $\frac{\partial n}{\partial T}\big|_{\mu} \geq 0$

$$\left.\frac{\partial N}{\partial \mu}\right|_{T} = \int_0^{+\infty} \mathrm{d}\epsilon\, \mathrm{d}(\epsilon) \left.\frac{\partial n}{\partial \mu}\right|_{T} \geq 0$$

> Because $\mathrm{d}(\epsilon) \geq 0$ and $\frac{\partial n}{\partial \mu}\big|_{T} \geq 0$

Meaning,

$$\left.\frac{\partial \mu}{\partial T}\right|_{N} < 0$$

(c) By introducing the fugacity $\varphi = e^{\beta\mu}$ as well as the function

$$f(\varphi) = \int_0^{\infty} \mathrm{d}x \frac{\sqrt{x}}{e^x/\varphi - 1}$$

determine graphically the chemical potential μ. What happens when the temperature is lowered? Show that there exists a critical temperature T_B, called the Bose temperature, for which $\mu = 0$. Note that

$$\int_0^{\infty} \mathrm{d}x \frac{\sqrt{x}}{e^x - 1} = \frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right)$$

where $\zeta(z)$ is the Riemann zeta function, which is defined for any complex number z such that $\mathrm{Re}(z) > 1$ by the Riemann series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. In particular, $\zeta(3/2) \approx 2.61$ and $\zeta(5/2) \approx 1.34$.

$$N = N(\mu) : \text{grand canonical}$$
$$\mu = \mu(N) : \text{canonical}$$

$$N = \int_0^{+\infty} \mathrm{d}\epsilon \, \mathrm{d}(\epsilon) \, n(\epsilon) = KV \int_0^{+\infty} \mathrm{d}\epsilon \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)} - 1}$$

We define the fugacity $\varphi = e^{\beta\mu} \in [0, 1)$,

$$\begin{aligned} N &= KV \int_0^{+\infty} \mathrm{d}\epsilon \frac{\sqrt{\epsilon}}{\frac{e^{\beta\epsilon}}{\varphi} - 1} \qquad\qquad X = \beta\epsilon, \quad \mathrm{d}X = \beta \mathrm{d}\epsilon \\ &= KV \int_0^{+\infty} \frac{\mathrm{d}X}{\beta^{3/2}} \frac{X^{1/2}}{\frac{e^X}{\varphi} - 1} \end{aligned}$$

We define $f(\varphi)$,

$$f(\varphi) \equiv \int_0^{+\infty} \mathrm{d}X \frac{\sqrt{X}}{\frac{e^X}{\varphi} - 1}$$

And so we get,

$$f(\varphi) = \frac{N}{KV(k_B T)^{3/2}}$$

Let's study $f(\varphi)$,

$$f(0) = 0 \qquad\qquad f(1) = \frac{\sqrt{\pi}}{2} \zeta\left(\frac{3}{2}\right)$$

- $\varphi \ll 1$

$$f(\varphi) = \int_0^{+\infty} \mathrm{d}X \sqrt{X} e^{-X} \varphi$$

- $\varphi \to 1^-$
  Let ε be a small parameter,

$$\varepsilon = 1 - \varphi \ll 1$$

$$
\begin{aligned}
f(\varphi) &= \int_0^{+\infty} \frac{\mathrm{d}X\sqrt{X}}{\frac{e^X}{1-\varepsilon} - 1} \\
&\approx \int_0^{+\infty} \frac{\mathrm{d}X\sqrt{X}}{e^X(1+\varepsilon) - 1} \\
&=?
\end{aligned}
$$

It does seems non-analytical.

So we will study its derivative now,

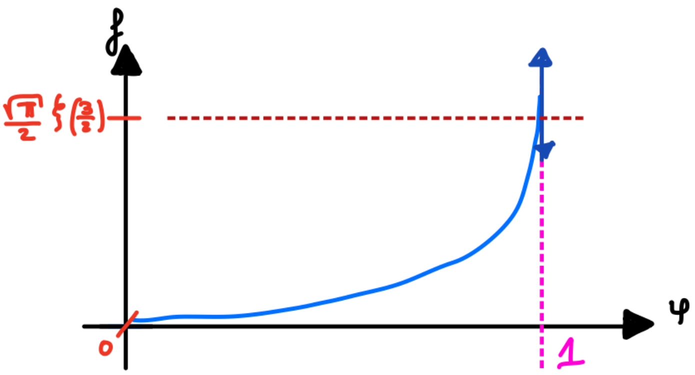
$$
f'(\varphi) = \int_0^{+\infty} \mathrm{d}X \frac{\sqrt{X}e^X}{(e^X - \varphi)^2} > 0
$$

$$
f'(\varphi = 1) = \int_0^{+\infty} \mathrm{d}X \frac{\sqrt{X}e^X}{(e^X - 1)^2} \propto \frac{1}{X^{3/2}}
$$

Meaning that

$$
f'(\varphi = 1) = +\infty
$$

Now we can make the schematic,



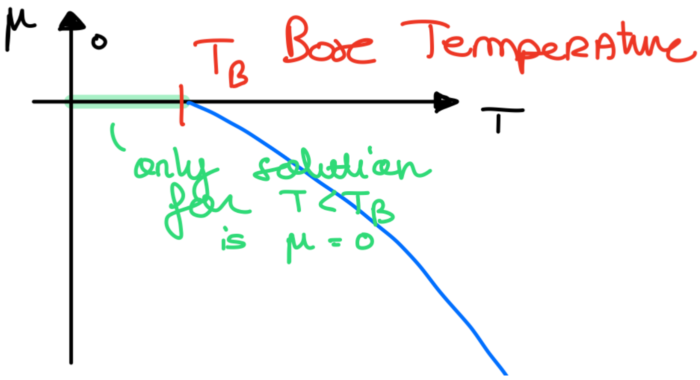
Remark :

$$
N = \sum_{\lambda} n_\lambda \approx \int_0^{+\infty} \mathrm{d}\epsilon\, \mathrm{d}(\epsilon)\, n(\epsilon)
$$

This approximation is only valid if the difference between two neighboring values is not too large.

$$n(\epsilon_0 = 0) = \frac{1}{e^{-\beta\mu} - 1} \gg 1$$

for $T < T_B$ where T_B is the Bose temperature and ϵ_0 is the energy of the ground state. This way we get a macroscopic number of particles in the ground state.



(d) We now consider that $T < T_B$ and we assume N to be fixed. Show that the number of particles in the ground state is given by

$$N_0 = N \left[1 - \left(\frac{T}{T_B}\right)^{3/2}\right]$$

Is it possible to condensate photons ?

$$\begin{aligned} N = \sum_{\lambda} n_\lambda &= N_0 + \sum_{\lambda \geq 1} n_\lambda \\ &\approx N_0 + \int_{\epsilon_1}^{+\infty} \mathrm{d}\epsilon \, \mathrm{d}(\epsilon) \, n(\epsilon) \\ &\approx N_0 + \int_{0}^{+\infty} \mathrm{d}\epsilon \, \mathrm{d}(\epsilon) \, n(\epsilon) \end{aligned}$$

$T < T_B : \mu = 0$,

$$\begin{aligned} N &= N_0 + KV \int_0^{+\infty} \mathrm{d}\epsilon \frac{\sqrt{\epsilon}}{e^{\beta\epsilon} - 1} \\ &= N_0 + N \left(\frac{T}{T_B}\right)^{3/2} \end{aligned}$$

Meaning,

$$N_0 = N \left[1 - \left(\frac{T}{T_B}\right)^{3/2}\right]$$

(e) Give an expression of the average energy E of the system in terms of an integral over ϵ that we will not explicitly calculate. Show that the grand-canonical potential Ω reads

$$\Omega = KVk_BT \int_0^\infty \mathrm{d}\epsilon \sqrt{\epsilon} \ln\left(1 - e^{-\beta(\epsilon-\mu)}\right)$$

Deduce from the two previous results that $\Omega = -2E/3$. Then, show that the pressure of the Bose gas is given by $P = 2E/3V$.

$$\begin{aligned}\bar{E} &= \sum_\lambda \epsilon_\lambda n_\lambda \\ &= \epsilon_0 n_0 + \sum_{\lambda \neq 0} \epsilon_\lambda n_\lambda\end{aligned}$$

$$\begin{aligned}\bar{E} &\approx \int_0^{+\infty} \mathrm{d}\epsilon\, \mathrm{d}(\epsilon)\, n(\epsilon)\epsilon \\ &= KV \int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)} - 1}\end{aligned}$$

We can perform the calculation for the grand potential,

$$\Omega = -k_BT \ln(\Xi)$$

$$\begin{aligned}\Xi &= \sum_{n_{\lambda_0}=0}^{+\infty} \cdots \exp\left[-\beta\left(\sum_\lambda \epsilon_\lambda n_\lambda - \mu \sum_\lambda n_\lambda\right)\right] \\ &= \sum_{n_{\lambda_0}=0}^{+\infty} \cdots \prod_\lambda \exp[-\beta\,(\epsilon_\lambda - \mu)\, n_\lambda] \\ &= \prod_\lambda \sum_{n_{\lambda_0}=0} [\exp(-\beta(\epsilon_\lambda - \mu))]^{n_\lambda} \\ &= \prod_\lambda \frac{1}{1 - \exp(-\beta(\epsilon_\lambda - \mu))}\end{aligned}$$

Meaning,

$$\begin{aligned}
\Omega &= k_B T \sum_\lambda \ln[1 - e^{-\beta(\epsilon_\lambda - \mu)}] \\
&= k_B T \ln[1 - e^{\beta\mu}] + k_B T \int_{\epsilon_1}^{+\infty} \mathrm{d}\epsilon \, \mathrm{d}(\epsilon) \ln[1 - e^{-\beta(\epsilon-\mu)}] \\
&\approx k_B T \int_0^{+\infty} \mathrm{d}\epsilon \, \mathrm{d}(\epsilon) \ln[1 - e^{-\beta(\epsilon-\mu)}] \\
&= k_B T \int_0^{+\infty} \mathrm{d}\epsilon K V \epsilon^{1/2} \ln[1 - e^{-\beta(\epsilon-\mu)}] \\
&= k_B T K V \frac{2}{3}\epsilon^{3/2} \ln[1 - e^{-\beta(\epsilon-\mu)}] \Big|_0^{+\infty} - KV \int_0^{+\infty} \mathrm{d}\epsilon k_B T \frac{2}{3}\epsilon^{3/2} \frac{\beta e^{-\beta(\epsilon-\mu)}}{1 - e^{-\beta(\epsilon-\mu)}} \\
&= -KV \int_0^{+\infty} \mathrm{d}\epsilon \frac{2}{3}\epsilon^{3/2} \frac{1}{-1 + e^{\beta(\epsilon-\mu)}}
\end{aligned}$$

Meaning,

$$\Omega = -\frac{2}{3}E$$

$$p = -\frac{\partial \Omega}{\partial V} = \frac{2}{3}\frac{\partial E}{\partial V} = \frac{2}{3}\frac{E}{V}$$

(f) Derive an expression for the pressure of the system at $T < T_B$. Note that

$$\int_0^\infty \mathrm{d}x \frac{x^{3/2}}{e^x - 1} = \frac{3\sqrt{\pi}}{4}\zeta\left(\frac{5}{2}\right)$$

$$\begin{aligned}
p &= \frac{2}{3}K \int_0^{+\infty} \mathrm{d}\epsilon \frac{\epsilon^{3/2}}{e^{\beta\epsilon} - 1} \qquad\qquad X = \beta\epsilon \\
&= \frac{2}{3}K (k_B T)^{5/2} \int_0^{+\infty} \mathrm{d}X \frac{X^{3/2}}{e^X - 1} \\
&= \frac{\zeta\left(\frac{5}{2}\right) g_s}{8\pi^{3/2}} \left(\frac{2m}{\hbar^2}\right)^{3/2} (k_B T)^{5/2}
\end{aligned}$$

No dependence of the volume.

(g) We now consider that T is kept constant, instead of V (N remains fixed throughout). What happens when the volume of the system is decreased ? Show that the Bose condensation takes place for

$$V_B = \frac{1}{(2s+1)\zeta(3/2)} N \Lambda_T^3$$

where $\Lambda_T = (2\pi\hbar^2/mk_BT)^{1/2}$ is the thermal de Broglie wavelength. Plot a few isothermal curves in a $P - V$ diagram. Discuss your results.

$$\frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right) = \frac{N}{KV_B(k_BT)^{3/2}}$$

Meaning,

$$V_B(T) = \frac{1}{g_s\zeta\left(\frac{3}{2}\right)}\Lambda_T^3$$

Where Λ_T^3 is the De Broglie thermal length.

(h) Liquid ^{4}He presents a superfluid transition at 2.17K. Compare such an experimental result to the Bose temperature. Parameters for liquid ^{4}He are : spin $s = 0$, density 0.12 g/cm^3, and $m = 4 \times m_{\text{proton}} = 6.7 \times 10^{-27}$ kg. We recall that $\hbar = 1.0 \times 10^{-34}$ J.s and $k_B = 1.4 \times 10^{-23}$ J/K.

$$T_B = \frac{2\pi\hbar^2}{k_Bm}\left[\frac{n}{g_s\zeta\left(\frac{3}{2}\right)}\right]^{2/3} = 2.4 \text{ K}$$